Solutions

1. (a) Observe that $f(x) \geq 0$ for all $x \in \mathbb{R}$ and that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^2\,\mathrm{d}x = \frac{1}{7}x^3\Big|_1^2 = \frac{8}{7} - \frac{1}{7} = 1$$

so that f is, in fact, a legitimate density.

1. (b) By definition,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^3\,\mathrm{d}x = \frac{3}{28}x^4\Big|_1^2 = \frac{3}{28}(16-1) = \frac{45}{28}.$$

1. (c) We find

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{3}{7}x^4\,\mathrm{d}x = \frac{3}{35}x^5\Big|_1^2 = \frac{3}{35}(32-1) = \frac{93}{35}$$

so that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{93}{35} - \left[\frac{45}{28}\right]^2 = \frac{2037}{27440} \doteq 0.0742347.$$

1. (d) Note that if $x < 1$, then $F(x) = 0$, while if $x > 2$, then $F(x) = 1$. However, if $1 \leq x \leq 2$, then

$$F(x) = \int_1^x \frac{3}{7}u^2\,\mathrm{d}u = \frac{1}{7}u^3\Big|_1^x = \frac{x^3}{7} - \frac{1}{7}.$$

In summary,

$$F(x) = \begin{cases} 0, & x < 1, \\ \dfrac{x^3}{7} - \dfrac{1}{7}, & 1 \leq x \leq 2, \\ 1, & x > 2. \end{cases}$$

1. (e) The median of X is that value a for which

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x = \frac{1}{2},$$

or equivalently, that value of a for which $F(a) = \mathbf{P}\,(X \leq a) = 1/2$. Thus, since we found F in **(d)**, we conclude that a satisfies

$$\frac{a^3}{7} - \frac{1}{7} = \frac{1}{2}$$

and so

$$a = \mathrm{med}(X) = \left(\frac{9}{2}\right)^{1/3}.$$

2. (a) If $x \geq 0$, then

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = xe^{-x}.$$

2. (b) If $y \geq 0$, then

$$\begin{aligned}
\mathbf{P}(Y_1 > y) = \mathbf{P}(\min\{X_1, X_2\} > y) = \mathbf{P}(X_1 > y, X_2 > y) &= \mathbf{P}(X_1 > y)\,\mathbf{P}(X_2 > y) \\
&= [1 - \mathbf{P}(X_1 \leq y)][1 - \mathbf{P}(X_2 \leq y)] \\
&= [1 - F(y)]^2 \\
&= [ye^{-y} + e^{-y}]^2 \\
&= (y+1)^2 e^{-2y}
\end{aligned}$$

so that

$$F_{Y_1}(y) = \mathbf{P}(Y_1 \leq y) = 1 - \mathbf{P}(Y_1 > y) = 1 - (y+1)^2 e^{-2y}.$$

Thus,

$$f_{Y_1}(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_1}(y) = 2y(y+1)e^{-2y}$$

for $y \geq 0$.

2. (c) If $y \geq 0$, then

$$\begin{aligned}
F_{Y_2}(y) = \mathbf{P}(Y_2 \leq y) = \mathbf{P}(\max\{X_1, X_2\} \leq y) &= \mathbf{P}(X_1 \leq y, X_2 \leq y) \\
&= \mathbf{P}(X_1 \leq y)\,\mathbf{P}(X_2 \leq y) \\
&= [F(y)]^2 \\
&= [1 - ye^{-y} - e^{-y}]^2.
\end{aligned}$$

Thus,

$$f_{Y_2}(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_2}(y) = 2ye^{-y}(1 - ye^{-y} - e^{-y})$$

for $y \geq 0$.

2. (d) If $Z_1 = Y_1^3$, then for $z \geq 0$, the distribution function of Z_1 is

$$F_{Z_1}(z) = \mathbf{P}(Z_1 \leq z) = \mathbf{P}\left(Y_1^3 \leq z\right) = \mathbf{P}\left(Y_1 \leq z^{1/3}\right) = \int_{-\infty}^{z^{1/3}} f_{Y_1}(y)\,\mathrm{d}y$$

so by the fundamental theorem of calculus, if $z \geq 0$, then

$$f_{Z_1}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{Z_1}(z) = f_{Y_1}(z^{1/3}) \frac{\mathrm{d}}{\mathrm{d}z} z^{1/3} = 2z^{1/3}(z^{1/3}+1)e^{-2z^{1/3}} \cdot \frac{1}{3} z^{-2/3} = \frac{2}{3}(1 + z^{-1/3})e^{-2z^{1/3}}.$$

2. (e) If $Z_2 = \sqrt{Y_2}$, then for $z \geq 0$, the distribution function of Z_2 is

$$F_{Z_2}(z) = \mathbf{P}(Z_2 \leq z) = \mathbf{P}\left(\sqrt{Y_2} \leq z\right) = \mathbf{P}\left(Y_2 \leq z^2\right) = \int_{-\infty}^{z^2} f_{Y_2}(y)\,\mathrm{d}y$$

so by the fundamental theorem of calculus, if $z \geq 0$, then

$$f_{Z_2}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{Z_2}(z) = f_{Y_2}(z^2) \frac{\mathrm{d}}{\mathrm{d}z} z^2 = 2z^2 e^{-z^2}(1 - z^2 e^{-z^2} - e^{-z^2}) \cdot 2z = 4z^3 e^{-z^2}(1 - z^2 e^{-z^2} - e^{-z^2}).$$

3. By the law of total probability,

$$\begin{aligned}
\mathbf{P}(X < Y) = \int_{-\infty}^{\infty} \mathbf{P}(Y > x) f_X(x)\,\mathrm{d}x &= \int_0^{\infty} \left[\int_x^{\infty} e^{-y}\,\mathrm{d}y \right] x e^{-x}\,\mathrm{d}x \\
&= \int_0^{\infty} \left[e^{-x} \right] x e^{-x}\,\mathrm{d}x \\
&= \int_0^{\infty} x e^{-2x}\,\mathrm{d}x \\
&= \frac{1}{4} \int_0^{\infty} u e^{-u}\,\mathrm{d}u \\
&= \frac{1}{4}.
\end{aligned}$$

(Note that this final integral equals 1 since it represents the total area under a density curve—the density for X, in fact.)

4. By the law of the unconscious statistician, we have

$$\mathbb{E}[F(X)] = \int_{-\infty}^{\infty} F(x) f(x)\,\mathrm{d}x.$$

If we make the change of variables $u = F(x)$, then $\mathrm{d}u = F'(x)\,\mathrm{d}x$. But we know that $F' = f$ so that $\mathrm{d}u = f(x)\,\mathrm{d}x$. Now for the limits of integration. Since $F(x) \to 1$ as $x \to \infty$ and since $F(x) \to 0$ as $x \to -\infty$, we find

$$\int_{-\infty}^{\infty} F(x) f(x)\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}u = \left. \frac{u^2}{2} \right|_0^1 = \frac{1}{2}$$

as required.