Math/Stat 251 Fall 2015
A Function of a Random Variable (November 2, 2015)

Let X be a continuous random variable with density function f. Sometimes we are interested in a function of a random variable. For instance, we might view X as a physical measurement and $g(X)$ as that measurement in different units. We've seen that $\mathbb{E}[g(X)]$, the mean or expected value of $g(X)$, is given by

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x$$

(which is sometimes called the law of the unconscious statistician). However, as we will now see, in many cases we can actually determine the distribution of $g(X)$.

Remark. The *pivot method* is a technique from statistical inference for constructing confidence intervals that requires one to do exactly this.

The basic technique is to determine the distribution function of $Y = g(X)$ from first principles. The density function of $Y = g(X)$ can then be found by differentiation.

Example. Suppose that $X \sim \text{Exp}(\lambda)$. Determine the distribution/density of $Y = e^X$.

Solution. If $X \sim \text{Exp}(\lambda)$, then $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $Y = e^X$. By definition,

$$\begin{aligned} F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}\left(e^X \leq y\right) = \mathbf{P}(X \leq \log y) = \int_{-\infty}^{\log y} f_X(x)\,\mathrm{d}x &= \int_0^{\log y} \lambda e^{-\lambda x}\,\mathrm{d}x \\ &= -e^{-\lambda x}\Big|_0^{\log y} \\ &= 1 - e^{-\lambda \log y} \\ &= 1 - y^{-\lambda} \end{aligned}$$

provided that $y \geq 1$. (Why is this the restriction on y? If $x \geq 0$ and $y = e^x$, then $y \geq e^0 = 1$.) We now find $f_Y(y)$.

- Method #1: direct differentiation of the distribution function

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}(1 - y^{-\lambda}) = \lambda y^{-1-\lambda}.$$

- Method #2: "symbolic" differentiation of the distribution function

$$\begin{aligned} f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\log y} \lambda e^{-\lambda x}\,\mathrm{d}x &= \lambda e^{-\lambda \log y} \cdot \frac{\mathrm{d}}{\mathrm{d}y}(\log y) \quad \text{by the chain rule} \\ &= \lambda y^{-\lambda} \cdot \frac{1}{y} \quad \text{as above.} \end{aligned}$$

Remark. We put subscripts on the density functions to keep track of the random variables. That is, f_X is the density function of X and f_Y is the density function of Y. We cannot use just f here since there are two different density functions being considered. The same is true for the distribution functions.

Remark. We observe that Method #2 can be generalized to any strictly increasing function g provided that its derivative g' exists.

Theorem and Proof. Suppose that X is a continuous random variables with density f_X and g is a strictly increasing, differentiable function. If $Y = g(X)$, then

- $F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(X \le g^{-1}(y)) = F_X(g^{-1}(y))$, and
- $f_Y(y) = \dfrac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \dfrac{\mathrm{d}}{\mathrm{d}y} \displaystyle\int_{-\infty}^{g^{-1}(y)} f_X(x)\,\mathrm{d}x = f_X(g^{-1}(y)) \cdot \dfrac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)$.

On the other hand, if g is strictly decreasing, then

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y).$$

(The extra minus sign is needed since $\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) < 0$.)

Summary. If g is strictly monotone, then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right|.$$

Remark. When you need to change variables, don't try to just plug into a memorized formula. Instead, follow either "Method #1" or "Method #2" directly.

Example. Suppose that X is a continuous random variable with density

$$f(x) = \frac{3}{7} x^2$$

for $1 \le x \le 2$. Determine the density function of $Y = 1/X^2$.

Solution. By definition,

$$\begin{aligned} F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\left(1/X^2 \le y\right) = \mathbf{P}\left(1/y \le X^2\right) = \mathbf{P}\left(X \ge y^{-1/2}\right) &= \int_{y^{-1/2}}^{\infty} f(x)\,\mathrm{d}x \\ &= \int_{y^{-1/2}}^{2} \frac{3}{7} x^2\,\mathrm{d}x \\ &= \frac{8}{7} - \frac{y^{-3/2}}{7} \end{aligned}$$

provided that $1/4 \le y \le 1$. Hence,

$$F_Y(y) = \begin{cases} 0, & y < 1/4, \\ \dfrac{8}{7} - \dfrac{y^{-3/2}}{7}, & 1/4 \le y \le 1, \\ 1, & y \ge 1 \end{cases}$$

and so

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{8}{7} - \frac{y^{-3/2}}{7} \right) = \frac{3}{14} y^{-5/2}$$

for $1/4 \le y \le 1$.