

Math/Stat 251 Fall 2015

The Maximum and Minimum of Two IID Random Variables (October 30, 2015)

Suppose that X_1 and X_2 are independent and identically distributed (iid) continuous random variables. By *independent*, we mean that

$$\mathbf{P}(X_1 \in A, X_2 \in B) = \mathbf{P}(X_1 \in A) \mathbf{P}(X_2 \in B)$$

for any $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. By *identically distributed* we mean that X_1 and X_2 each have the same distribution function F (and therefore the same density function f).

Two quantities of interest are the maximum and minimum of X_1 and X_2 . It turns out to be surprisingly easy to determine the distribution and density functions of the maximum and minimum.

The two key observations are that

$$\max\{X_1, X_2\} \leq x \text{ if and only if both } X_1 \leq x \text{ and } X_2 \leq x$$

and

$$\min\{X_1, X_2\} > x \text{ if and only if both } X_1 > x \text{ and } X_2 > x.$$

In other words, an upper bound for the maximum gives an upper bound for each of X_1 and X_2 , while a lower bound for the minimum gives a lower bound for each of X_1 and X_2 .

Distribution of $\max\{X_1, X_2\}$

Suppose that $X = \max\{X_1, X_2\}$. By definition, the distribution function of X is

$$\begin{aligned} F_X(x) &= \mathbf{P}(X \leq x) = \mathbf{P}(\max\{X_1, X_2\} \leq x) = \mathbf{P}(X_1 \leq x \text{ and } X_2 \leq x) \\ &= \mathbf{P}(X_1 \leq x, X_2 \leq x) \\ &= \mathbf{P}(X_1 \leq x) \mathbf{P}(X_2 \leq x) \end{aligned}$$

using the fact that X_1 and X_2 are independent. However, both X_1 and X_2 have the *same* distribution function F and the *same* density function f . This means that

$$\mathbf{P}(X_1 \leq x) = F(x) = \int_{-\infty}^x f(t) dt \quad \text{and} \quad \mathbf{P}(X_2 \leq x) = F(x) = \int_{-\infty}^x f(t) dt.$$

Therefore,

$$F_X(x) = F(x) \cdot F(x) = [F(x)]^2.$$

The density function of $X = \max\{X_1, X_2\}$ can now be found by differentiation, namely

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} [F(x)]^2 = 2F(x)F'(x) = 2f(x)F(x).$$

Distribution of $\min\{X_1, X_2\}$

Suppose that $Y = \min\{X_1, X_2\}$. By definition, the distribution function of Y is

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(\min\{X_1, X_2\} \leq y).$$

However, knowing an upper bound on the minimum is not of any use to us. Instead, we consider

$$F_Y(y) = \mathbf{P}(Y \leq y) = 1 - \mathbf{P}(Y > y) = 1 - \mathbf{P}(\min\{X_1, X_2\} > y).$$

This *is* useful to us since

$$\mathbf{P}(\min\{X_1, X_2\} > y) = \mathbf{P}(X_1 > y, X_2 > y) = \mathbf{P}(X_1 > y)\mathbf{P}(X_2 > y)$$

using the fact that X_1 and X_2 are independent. However, both X_1 and X_2 have the *same* distribution function F and the *same* density function f . This means that

$$\mathbf{P}(X_1 > y) = 1 - F(y) = \int_y^\infty f(t) dt \quad \text{and} \quad \mathbf{P}(X_2 > y) = 1 - F(y) = \int_y^\infty f(t) dt.$$

Therefore,

$$F_Y(y) = 1 - \mathbf{P}(Y > y) = 1 - [1 - F(y)] \cdot [1 - F(y)] = 1 - [1 - F(y)]^2.$$

The density function of $Y = \min\{X_1, X_2\}$ can now be found by differentiation, namely

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - [1 - F(y)]^2) = 2[1 - F(y)]F'(y) = 2f(y)[1 - F(y)].$$

Example. Suppose that X_1 and X_2 are independent random variables each having the $\text{Exp}(\lambda)$ distribution. Determine the density functions of $\max\{X_1, X_2\}$, and $\min\{X_1, X_2\}$.

Solution. Since X_1 and X_2 are iid $\text{Exp}(\lambda)$ random variables, they have common density function $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, and common distribution functions

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Thus, the density function of $X = \max\{X_1, X_2\}$ is

$$f_X(x) = 2\lambda e^{-\lambda x}[1 - e^{-\lambda x}]$$

for $x \geq 0$, and the density function of $Y = \min\{X_1, X_2\}$ is

$$f_Y(y) = 2\lambda e^{-2\lambda y}$$

for $y \geq 0$. Note that we recognize the distribution of Y as $Y \sim \text{Exp}(2\lambda)$.