

- [9] 1. An urn contains 5 red balls, 10 green balls and 15 yellow balls. You randomly select 5 balls, without replacement.

(a) What is the probability that there are no red balls among the ones you picked?

(4 points) $\binom{25}{5} / \binom{30}{5} = 1265/3393 = .37283$

$P(\text{no red balls}) =$

(b) What is the probability that at least one of the three colours is not present among the balls you picked? *Hint: $P(A \cup B \cup C) = ?$*

(5 pts) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) = (\binom{25}{5} + \binom{20}{5} + \binom{15}{5} - \binom{5}{5} - \binom{10}{5} - \binom{15}{5}) / \binom{30}{5} = 68381/142506 = .47985$

Probability =

- [12] 2. A physician has five patients with Ross's disease. There are two possible treatments, A and B. The physician gives treatment A to three randomly selected patients, and treatment B to the other two. Suppose that treatment A produces a cure in any given patient with probability 0.3, while treatment B produces a cure with probability 0.6.

(a) Alice is one of the patients. What is the probability that Alice will be cured?
 (3 pts) $(3/5)(.3) + (2/5)(.6) = .42$

$P(C) =$

(b) Given that Alice is cured, what is the probability that she was given treatment A?
 (4 pts) $P(B|C) = P(BC)/P(C) = (3/5)(.3)/.42 = 3/7 = .42857$

$P(A|C) =$

(c) What is the probability that the two treatments will produce the same number of cures?
 (5 pts) $P(0,0) + P(1,1) + P(2,2) = (.7^3)(.4^2) + \binom{3}{1}\binom{2}{1}(.3)(.7)^2(.6)(.4) + \binom{3}{2}\binom{2}{2}(.3)^2(.7)(.6)^2 = .3346$

$P(N_A = N_B) =$

[20] **3.** Anne and Bob are playing a game. Each time they play, Anne's net gain is +\$4 with probability $4/35$, +\$1 with probability $2/7$, and -\$1 with probability $3/5$.

(a) Find the mean and variance of Anne's net gain from playing this game once.

(6 pts) $E[X] = 4 \times 4/35 + 1 \times 2/7 - 1 \times 3/5 = 1/7$, $E[X^2] = 4^2 \times 4/35 + 1 \times 2/7 + 1 \times 3/5 = 19/7 = 2.7143$, $\text{Var}(X) = 19/7 - 1/49 = 132/49 = 2.6939$

mean , variance

(b) Let S_{49} be Anne's net gain from playing the game 49 times. Using Chebyshev's inequality we can say that, for certain integers a and b , $P(S_{49} \leq a \text{ or } S_{49} \geq b) \leq 1/4$. What are the maximum possible a and minimum possible b ?

(5 pts) S_{49} has mean $\mu = 1/7 \times 49 = 7$ and standard deviation $\sqrt{49 \times 132/49} = \sqrt{132} = 11.489$. Chebyshev says $P(|S_{49} - \mu| \geq 2\sigma) \leq 1/4$. $\mu - 2\sigma = -15.978$ and $\mu + 2\sigma = 29.978$. Since $S_{49} \leq -15.978$ is equivalent to $S_{49} \leq -16$ and $S_{49} \geq 29.978$ is equivalent to $S_{49} \geq 30$, $a = -16$ and $b = 30$.

$a =$, $b =$

- (c) Use the Central Limit Theorem (with continuity correction) to approximate $P(S_{49} \leq a)$, using the a from your answer above.

(6 pts) $P(S_{49} \leq -16) = P(S_{49} \leq -15.5) \approx \Phi\left(\frac{-15.5 - 7}{11.489}\right) = \Phi(-1.9584) = 1 - \Phi(1.9584) = 1 - .9749 = .0251$ using the table (and interpolating linearly).

$P(S_{49} \leq a) \approx$

- (d) What does the Law of Large Numbers say will happen if Anne and Bob keep playing this game? Either the weak or strong version will do.
(3 pts) $\lim_{n \rightarrow \infty} S_n/n = 1/7$ with probability 1

[8] 4. A commercial airliner has probability 5×10^{-7} of suffering a serious accident on any given flight. There are 10^6 commencial airline flights each year in the US. Assume that accidents on different flights are independent.

(a) Name the probability distribution, and state the parameter value(s), for the number of serious accidents on commercial flights in the US in a year.

The distribution is with parameter(s)

(3 pts) *Binomial with $n = 10^6$ and $p = 5 \times 10^{-7}$.*

(b) Find a numerical approximation for the probability that there are at least two such accidents in a year.

$P(X \geq 2) \approx$ (5 pts) *This can be approximated using a Poisson*

distribution with $\lambda = np = 0.5$. $P(X \geq 2) = 1 - P(X \leq 1) \approx 1 - e^{-0.5}(1 + 0.5) = 0.090204$

[14] 5. A continuous random variable X has probability density function

$$f(x) = \begin{cases} a & \text{for } -2 < x < 0 \\ bx & \text{for } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants. It is known that $E(X) = 0$.

(a) Determine a and b .

(6 points) $2a + b/2 = 1$, $-2a + b/3 = 0$ so $a = 1/5$, $b = 6/5$.

$a =$, $b =$

(b) Find $\text{Var}(X)$.

(4 pts) $\text{Var}(X) = E[X^2] = 8a/3 + b/4 = 5/6$

$\text{Var}(X) =$

(c) Find the median of X , i.e. a number m such that $P(X \leq m) = 1/2$.

(4 pts) $\int_{-2}^0 f(x) dx = 2a = 2/5 < 1/2$ so $m > 0$. $\int_{-2}^m f(x) dx = 2/5 + (6/5)(m^2/2) = 1/2$ so $m = 1/\sqrt{6} = .40825$.

$m =$

[12] 6. Cars arrive at a certain gas station according to a Poisson process at a rate of 6 cars per hour. The gas station opens at 8:00 am.

(a) Find the variance of the number of cars that arrive between 8:00 am and 10:00 am.

Variance =

(2 pts) For a Poisson rv with parameter 12, the variance is 12.

(b) Find the probability that exactly 2 cars arrive between 8:00 am and 8:30 am.

Probability =

(4 pts) $N(1/2)$ is Poisson with parameter 3; $P(N(1/2) = 2) = (9/2)e^{-3} = 0.22404$

(c) Find the conditional probability that exactly 2 cars arrive between 8:00 am and 8:30 am, given that exactly 4 arrive between 8:00 am and 9:00 am.

(6 pts) $N(1/2)$ and $N(1) - N(1/2)$ are independent Poisson with parameter 3, so $P(N(1/2) = 2, N(1) = 4) = P(N(1/2) = 2)P(N(1) - N(1/2) = 2) = ((9/2)e^{-3})^2$; $P(N(1) = 4) = (6^4/24)e^{-6}$, $P(N(1/2) = 2|N(1) = 4) = ((9/2)e^{-3})^2 / ((6^4/24)e^{-6}) = 3/8 = 0.375$.

Conditional probability =

[14] 7. You arrive at the post office and as you enter, each of the two clerks, Jim and Jack, starts serving a client. The amount of time needed by Jim to serve his client is X minutes, and the amount of time needed by Jack to serve his client is Y minutes, where X and Y are independent exponential random variables, X having parameter (rate) $1/2$ per minute and Y having parameter $1/4$ per minute.

- (a) What is the probability that Jim will spend at least a minutes with the client he started serving when you arrived?

$P(X \geq a) =$ (2 pts) $P(X \geq a) = 1 - F_X(a) = e^{-a/2}$

- (b) Suppose, when you entered the post office, Jim had already been serving his client for 2 minutes. What would be the probability that he would spend at least an additional a minutes with this client? Explain.

(2 pts) By “lack of memory” property, still $e^{-a/2}$.

- (c) Find the expected value of $X - Y$

$E[X - Y] =$
 (3 pts) $E[X - Y] = E[X] - E[Y] = 2 - 4 = -2$

- (d) Since you are next in line, you will be served by whichever clerk is first to finish serving his current client. Find the probability that you will be served by Jim.

$P(X < Y) =$
 (7 pts) The joint density is $f(x, y) = (1/8)e^{-x/2}e^{-y/4}$. $P(X < Y) = \int_0^\infty dx \int_x^\infty dy f(x, y) = \int_0^\infty dx (1/2)e^{-3x/4} = 2/3$

[11] 8. Two discrete random variables X and Y have the joint probability mass function

$$P(X = x, Y = y) = p(x, y) = \begin{cases} 2^{1-x}3^{-1-y} & \text{for } x = 1, 2, 3, \dots, y = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal probability mass function of X , i.e. $p_X(x) = P(X = x)$, for $x = 1, 2, 3, \dots$

$p_X(x) =$

(4 pts) $2^{1-x} \sum_{y=0}^{\infty} 3^{-1-y} = (2/3)2^{-x}(1/(1 - 1/3)) = 2^{-x}$

(b) Are X and Y independent? Circle **Yes** or **No**.
(2 pts) **Yes**

(c) Find the moment generating function of $X + Y$.

$M_{X+Y}(t) =$

(5 pts) $2e^t / ((2 - e^t)(3 - e^t))$

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The University of British Columbia
Sessional Examinations - December 2010

Mathematics 302
Introduction to Probability
R. Israel (Section 101) and M. Kozdron (Section 102)

Closed book examination

Time: $2\frac{1}{2}$ hours

Name _____ Signature _____

Student Number _____ Section _____

Special Instructions:

Non-graphing, non-programmable calculators allowed; no other books, notes, electronic devices or other aids. Show your work. If you give an answer in decimals, use 5 significant digits. For questions that provide a box, put your answer (but not your work) there. A table of the standard normal distribution is on the last page, which can also be used for “rough work” that you don’t want marked

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2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
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 - Speaking or communicating with other candidates.
 - Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
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1		9
2		12
3		20
4		8
5		14
6		12
7		14
8		11
Total		100