Statistics 251–Introduction to Probability
Fall 2015 (201530)
Final Exam Solutions

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1. (a) By definition, $F_X(x) = \mathbf{P}(X \le x)$. Note that if $x < 1$, then $F_X(x) = 0$. If $1 \le x \le 2$, then

$$F_X(x) = \int_1^x (4 - 2t)\,\mathrm{d}t = (4t - t^2)\Big|_1^x = 4x - x^2 - 3.$$

If $x > 2$, then $F_X(x) = 1$. In summary,

$$F_X(x) = \begin{cases} 0, & x < 1, \\ 4x - x^2 - 3, & 1 \le x \le 2, \\ 1, & x \ge 2. \end{cases}$$

1. (b) We find

$$\mathbb{E}(X) = \int_1^2 (4x - 2x^2)\,\mathrm{d}x = \left(2x^2 - \frac{2}{3}x^3\right)\Big|_1^2 = 8 - \frac{16}{3} - 2 + \frac{2}{3} = 6 - \frac{14}{3} = \frac{4}{3}.$$

1. (c) We find

$$\mathbb{E}(X^2) = \int_1^2 (4x^2 - 2x^3)\,\mathrm{d}x = \left(\frac{4}{3}x^3 - \frac{2}{4}x^4\right)\Big|_1^2 = \frac{32}{3} - \frac{32}{4} - \frac{4}{3} + \frac{2}{4} = \frac{28}{3} - \frac{30}{4} = \frac{56}{6} - \frac{45}{6} = \frac{11}{6}$$

and so

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{11}{6} - \left(\frac{4}{3}\right)^2 = \frac{11}{6} - \frac{16}{9} = \frac{33}{18} - \frac{32}{18} = \frac{1}{18}.$$

1. (d) Note that $m = \mathrm{Med}(X)$ if $F_X(m) = 0.5$. That is, m satisfies

$$4m - m^2 - 3 = \frac{1}{2}, \quad \text{or equivalently,} \quad m^2 - 4m + \frac{7}{2} = 0.$$

Solving for m implies that

$$m \in \left\{\frac{4 \pm \sqrt{16 - 14}}{2}\right\} = \left\{2 - \frac{1}{\sqrt{2}},\ 2 + \frac{1}{\sqrt{2}}\right\}$$

Since $\mathbf{P}(1 \le X \le 2) = 1$, it is clear that the smaller of the two roots of the quadratic satisfied by m is the median; that is,

$$\mathrm{Med}(X) = 2 - \frac{1}{\sqrt{2}}.$$

1. (e) Note that $\mathbf{P}(X \ge 1) = 1$ implying that

$$\mathbf{P}(X \le 1.5 \mid X \ge 1) = \mathbf{P}(X \le 1.5) = F_X(1.5) = 4 \cdot \frac{3}{2} - \left(\frac{3}{2}\right)^2 - 3 = \frac{3}{4}.$$

2. Note that $XY = 0$ if and only if at least one of X or Y equals 0. That is,

$$\{XY = 0\} = \{X = 0\} \cup \{Y = 0\}$$

and so

$$\begin{aligned}\mathbf{P}(XY=0) = \mathbf{P}(\{X=0\}\cup\{Y=0\}) &= \mathbf{P}(X=0)+\mathbf{P}(Y=0)-\mathbf{P}(\{X=0\}\cap\{Y=0\})\\ &= \mathbf{P}(X=0)+\mathbf{P}(Y=0)-\mathbf{P}(X=0)\mathbf{P}(Y=0)\end{aligned}$$

where the last equality uses the fact that X and Y are independent.

3. Note that $B \subset A\cup B$ so that $0.4 = \mathbf{P}(B) \le \mathbf{P}(A\cup B)$. Moreover,

$$\mathbf{P}(A\cup B) = \mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B) \le \mathbf{P}(A)+\mathbf{P}(B) = 0.4+0.2 = 0.6$$

implying that $0.4 \le \mathbf{P}(A\cup B) \le 0.6$. (Note that there are no tighter bounds possible using only the given information. In particular, if you use $A\subset A\cup B$ to conclude $0.2 = \mathbf{P}(A) \le \mathbf{P}(A\cup B)$, then, although it is a true statement, it is not the largest possible lower bound using the given information.)

4. Let A_i be the event that Andrew opens the lock on the ith try. Therefore, the probability that Andrew opens the lock on the third try is

$$\mathbf{P}(A_1^c\cap A_2^c\cap A_3) = \mathbf{P}(A_1^c)\cdot\mathbf{P}(A_2^c \mid A_1^c)\cdot\mathbf{P}(A_3 \mid A_1^c\cap A_2^c) = \frac{9}{10}\cdot\frac{8}{9}\cdot\frac{1}{8} = \frac{1}{10}.$$

5. Let A denote Alice's number and let B denote Bob's number. Clearly $\mathbf{P}(A=B) = \frac{6}{36}$ implying that $\mathbf{P}(A\ne B) = \frac{30}{36}$. By symmetry, $\mathbf{P}(A<B) = \mathbf{P}(B<A) = \frac{15}{36}$. Therefore if X denotes Alice's winnings, then

$$\mathbb{E}(X) = 10\cdot\mathbf{P}(X=10) - 5\cdot\mathbf{P}(X=5) = 10\mathbf{P}(A>B) - 5\mathbf{P}(A\ge B) = 10\cdot\frac{15}{36} - 5\cdot\frac{21}{36} = \frac{45}{36} = \frac{5}{4}.$$

6. The distribution function of Y is

$$F_Y(y) = \mathbf{P}(Y\le y) = \mathbf{P}\left(2+\frac{2}{X}\le y\right) = \mathbf{P}\left(\frac{2}{X}\le y-2\right) = \mathbf{P}\left(X \ge \frac{2}{y-2}\right) = 1-\mathbf{P}\left(X\le\frac{2}{y-2}\right)$$

implying that the density function of Y is

$$\begin{aligned}f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = -f_X\left(\frac{2}{y-2}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}y}\frac{2}{y-2} &= \frac{2}{(y-2)^2}f_X\left(\frac{2}{y-2}\right)\\ &= \frac{2}{(y-2)^2}\cdot 168\cdot\left(\frac{2}{y-2}\right)^5\left(1-\frac{2}{y-2}\right)^2\\ &= 10752\frac{(y-4)^2}{(y-2)^9}\end{aligned}$$

provided that $y>4$.

7. Let $X_i = 1$ if the ith flip is heads and let $X_i = 0$ if the ith flip is tails. The total number of heads is $S = X_1+\cdots+X_{400}$. Note that $\mathbb{E}(X_i) = 1/2$ and $\mathrm{Var}(X_i) = 1/4$ so that

$$\mathbb{E}(S) = \mathbb{E}(X_1+\cdots+X_{400}) = \mathbb{E}(X_1)+\cdots+\mathbb{E}(X_{400}) = 400\cdot\frac{1}{2} = 200.$$

Furthermore, since $X_1,\ldots,X_{400}$ are independent,

$$\mathrm{Var}(S) = \mathrm{Var}(X_1+\cdots+X_{400}) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_{400}) = 400\cdot\frac{1}{4} = 100.$$

Therefore, Chebychev's inequality implies

$$\mathbf{P}(185 \le S \le 215) = \mathbf{P}(-15 \le S - 200 \le 15) = \mathbf{P}(|S - 200| \le 15) \ge 1 - \frac{\mathrm{Var}(S)}{15^2} = 1 - \frac{100}{225} = \frac{5}{9}.$$

8. Using the fact that X and Y are independent, we conclude that the moment generating function for Z is

$$\begin{aligned} m_Z(t) = \mathbb{E}(e^{tZ}) = \mathbb{E}(e^{t(X+Y)}) = \mathbb{E}(e^{tX}e^{tY}) = \mathbb{E}(e^{tx})\mathbb{E}(e^{tY}) = m_X(t)m_Y(t) &= (1-2t)^{-3}(1-2t)^{-2} \\ &= (1-2t)^{-5}. \end{aligned}$$

Since

$$m_Z'(t) = 10(1-2t)^{-6}, \quad m_Z''(t) = 120(1-2t)^{-7}, \quad m_Z'''(t) = 1680(1-2t)^{-8}$$

we conclude that

$$\mathbb{E}(Z^3) = m_Z'''(0) = 1680.$$

9. (a) The probability that motor i operates for at least 5 months is

$$P(X_i \ge 5) = \int_5^\infty e^{-x}\,\mathrm{d}x = e^{-5}.$$

Therefore, the probability that at least 3 motors operate for at least 5 months is

$$\begin{aligned} &\mathbf{P}(\text{at least 3 of the 5 motors operate for at least 5 months}) \\ &= \mathbf{P}(\text{exactly 3 of the 5 motors operate for at least 5 months}) \\ &\quad + \mathbf{P}(\text{exactly 4 of the 5 motors operate for at least 5 months}) \\ &\qquad + \mathbf{P}(\text{exactly 5 of the 5 motors operate for at least 5 months}) \\ &= \binom{5}{3}(e^{-5})^3(1-e^{-5})^2 + \binom{5}{4}(e^{-5})^4(1-e^{-5})^1 + \binom{5}{5}(e^{-5})^5(1-e^{-5})^0 \\ &= 10e^{-15}(1-e^{-5})^2 + 5e^{-20}(1-e^{-5}) + e^{-25} \\ &= e^{-25}(10e^{-10} - 15e^{-5} - 6) \\ &\doteq 0.0000030281 \end{aligned}$$

9. (b) By definition, the distribution function of Y is $F_Y(y) = \mathbf{P}(Y \le y)$. Clearly, if $y \le 0$, then $F_Y(y) = 0$. However, if $y > 0$, then

$$\begin{aligned} \mathbf{P}(Y \le y) = 1 - \mathbf{P}(Y > y) &= 1 - \mathbf{P}(\min\{X_1, X_2, X_3, X_4, X_5\} > y) \\ &= 1 - \mathbf{P}(X_1 > y, X_2 > y, X_3 > y, X_4 > y, X_5 > y) \\ &= 1 - \mathbf{P}(X_1 > y)\mathbf{P}(X_2 > y)\mathbf{P}(X_3 > y)\mathbf{P}(X_4 > y)\mathbf{P}(X_5 > y) \\ &= 1 - [\mathbf{P}(X_1 > y)]^5 \\ &= 1 - (e^{-y})^5 \\ &= 1 - e^{-5y}. \end{aligned}$$

In summary,

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ 1 - e^{-5y}, & y > 0. \end{cases}$$

Thus, the density function of Y is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \begin{cases} 0, & y \le 0, \\ 5e^{-5y}, & y > 0, \end{cases}$$

and so

$$\mathbb{E}(Y) = \int_0^\infty 5ye^{-5y}\,\mathrm{d}y = \frac{1}{5}\int_0^\infty ue^{-u}\,\mathrm{d}u = \frac{1}{5}\cdot\Gamma(2) = \frac{1}{5}.$$

10. We observe that $\mathbf{P}(0 \le X \le 1) = 1$ and so if $Y > 1$, then Y is necessarily greater than X. That is, $\mathbf{P}(X > Y | Y > 1) = 0$. Hence, using the law of total probability, we obtain

$$\begin{aligned}\mathbf{P}(X > Y) &= \int_{-\infty}^\infty \mathbf{P}(X > Y | Y = y) f_Y(y)\,\mathrm{d}y = \int_{-\infty}^\infty \mathbf{P}(X > y) f_Y(y)\,\mathrm{d}y = \frac{1}{2}\int_0^2 \mathbf{P}(X > y)\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 \mathbf{P}(X > y)\,\mathrm{d}y + \frac{1}{2}\int_1^2 \mathbf{P}(X > y)\,\mathrm{d}y = \frac{1}{2}\int_0^1 \mathbf{P}(X > y)\,\mathrm{d}y + 0 = \frac{1}{2}\int_0^1 \left[\int_y^1 \mathrm{d}x\right]\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 (1-y)\,\mathrm{d}y = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.\end{aligned}$$

11. (a) If X_t denotes the number of pairs of shoes that Jessica has bought up to time t (measured in weeks), then X_t has a Poisson distribution with parameter λt. Since she is assumed to buy shoes at a rate of 1 pair per week, we know $\lambda = 1$. Therefore, the expected number of shoes that she buys in one year is $\mathbb{E}(X_{52}) = 52$.

11. (b) The probability that she bought 6 pairs of shoes in February 2015 is

$$\mathbf{P}(X_4 = 6) = \frac{4^6}{6!}e^{-4}.$$

11. (c) Using properties of conditional probability and the Poisson process, we find

$$\begin{aligned}\mathbf{P}(X_1 = 2 | X_4 = 6) = \frac{\mathbf{P}(X_4 = 6 | X_1 = 2)\mathbf{P}(X_1 = 2)}{\mathbf{P}(X_4 = 6)} = \frac{\mathbf{P}(X_3 = 4)\mathbf{P}(X_1 = 2)}{\mathbf{P}(X_4 = 6)} &= \frac{\frac{3^4}{4!}e^{-3}\frac{1^2}{2!}e^{-1}}{\frac{4^6}{6!}e^{-4}} \\ &= \frac{1215}{4096}.\end{aligned}$$