University of Regina
Department of Mathematics & Statistics
Final Examination
201530

Mathematics/Statistics 251
Introduction to Probability

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This exam has 11 problems and 2 numbered pages.

This exam is worth 150 points. The number of points per problem is indicated. For problems with multiple parts, all parts are equally weighted.

You have 3 hours to complete this exam. Please read all instructions carefully, and check your answers. Write your solutions in the exam booklets. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You are permitted to use one $8\frac{1}{2} \times 11$ *double sided page of handwritten notes. You are also permitted to use a calculator on this exam. No other aids are allowed. A cellular telephone is NOT considered a calculator and is not permitted.*

You must hand in all exam booklets containing solutions, although you may keep the problems.

1. (*25 points*) Suppose that the continuous random variable X has density function $f_X(x)$ given by

$$f_X(x) = \begin{cases} 4-2x, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine $F_X(x)$, the distribution function of X.

(b) Compute $\mathbb{E}(X)$, the expected value of X.

(c) Compute $\mathrm{Var}(X)$, the variance of X.

(d) Compute $\mathrm{Med}(X)$, the median of X. Recall that $\mathrm{Med}(X) = m$ if $\mathbf{P}(X \le m) = 0.5$.

(e) Compute $\mathbf{P}(X \le 1.5 \mid X \ge 1)$.

2. (*8 points*) Suppose that X and Y are independent discrete random variables. Carefully prove that $\mathbf{P}(XY = 0) = \mathbf{P}(X = 0) + \mathbf{P}(Y = 0) - \mathbf{P}(X = 0)\mathbf{P}(Y = 0)$.

3. (*8 points*) It is known that A and B are events with $\mathbf{P}(A) = 0.2$ and $\mathbf{P}(B) = 0.4$. Based on this information, determine the largest number a and the smallest number b such that $a \le \mathbf{P}(A \cup B) \le b$.

4. (*10 points*) Andrew is an absent minded janitor. He has 10 keys on his keyring, but has forgotten which key opens the lock on the storage room door. Suppose that Andrew tries the keys successively in the lock on the storage room door until it opens. What is the probability that the third key Andrew tries will open the lock?

5. (*12 points*) Alice and Bob are playing the following game. Each of them independently rolls a standard six-sided die. If Alice's number is *strictly larger* than Bob's number, then she wins \$10 from Bob. If Alice's number is *not* strictly larger than Bob's number, then she loses \$5 to Bob. Determine the amount of money that Alice expects to win after playing this game once.

6. (*12 points*) Suppose that the density function of the continuous random variable X is

$$f_X(x) = \begin{cases} 168x^5(1-x)^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

If the random variable Y is defined as $Y = 2 + \dfrac{2}{X}$, determine $f_Y(y)$, the density of Y.

7. (*12 points*) You flip a fair coin 400 times, independently from one flip to the next. Use Chebyshev's inequality to find a lower bound for the probability that the number of heads you will see is between 185 and 215 (both values included).

8. (*12 points*) An investment banker's portfolio contains two risky assets. The value (in thousands of dollars) of the first asset is a random variable X whose moment generating function is

$$m_X(t) = (1-2t)^{-3}.$$

The value (in thousands of dollars) of the second asset is a random variable Y whose moment generating function is

$$m_Y(t) = (1-2t)^{-2}.$$

The total value (in thousands of dollars) of the investment banker's portfolio is $Z = X + Y$. If the values of the two risky assets X and Y are independent, compute $\mathbb{E}(Z^3)$.

9. (*24 points*) A complex machine is able to run effectively provided that at least 3 of its 5 motors are operating.

It is known that motor i, for $i = 1, 2, 3, 4, 5$, operates for a random time X_i (measured in months). It is also known that the random times X_1, X_2, X_3, X_4, X_5 are independent and identically distributed random variables having common density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

(a) Determine the probability that the complex machine runs effectively for at least 5 months.

(b) The random time (in months) until the first motor fails is $Y = \min\{X_1, X_2, X_3, X_4, X_5\}$. Compute $\mathbb{E}(Y)$.

10. (*15 points*) Suppose that the random variable X is uniformly distributed on the interval $[0, 1]$ and that the random variable Y is uniformly distributed on the interval $[0, 2]$ so that the density functions of X and Y are

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{2}, & 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose further that X and Y are independent. Compute $\mathbf{P}(X > Y)$.

11. (*12 points*) Jessica is a compulsive shoe shopper. Suppose that Jessica buys shoes according to a Poisson process at a rate of 1 pair of shoes per week.

(a) How many pairs of shoes does Jessica expect to buy in one year?

(b) What is the probability that Jessica bought 6 pairs of shoes in February 2015? (Note that there were 28 days in February 2015.)

(c) If Jessica bought 6 pairs of shoes in February 2015, what is the probability that she bought exactly 2 of those pairs during the first week?