Math/Stat 251 Fall 2015
Solutions to Assignment #6

1. (a) Suppose that A is the event $A = \{\text{John smoked marijuana}\}$, and suppose further that B is the event $B = \{\text{John said yes}\}$. The law of total probability implies that

$$\mathbf{P}(B) = \mathbf{P}(B \mid A)\,\mathbf{P}(A) + \mathbf{P}(B \mid A^c)\,\mathbf{P}(A^c).$$

We do not know whether or not John said yes. We only know that $40/100 = 0.40$ of respondents said yes. Therefore, $\mathbf{P}(B) = 0.4$. We also know that if John has, in fact, smoked marijuana, then he would have answered truthfully (i.e., said yes) provided that he rolled a 1, 2, 3, or 4. That is, $\mathbf{P}(B \mid A) = 4/6$. Similarly, if we know that John has not smoked marijuana, then he would have lied (i.e., said yes) provided that he rolled a 5 or 6. That is, $\mathbf{P}(B \mid A^c) = 2/6$. In other words, we find

$$0.4 = \frac{4}{6}\mathbf{P}(A) + \frac{2}{6}\mathbf{P}(A^c).$$

Of course, $\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$ implying that

$$0.4 = \frac{4}{6}\mathbf{P}(A) + \frac{2}{6}(1 - \mathbf{P}(A))$$

and so

$$\mathbf{P}(\text{John smoked marijuana}) = \mathbf{P}(A) = \frac{0.4 - 2/6}{4/6 - 2/6} = 0.2.$$

(b) By Bayes' rule, the required probability is

$$\mathbf{P}(\text{John smoked marijuana} \mid \text{John said yes}) = \mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\,\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{4/6 \cdot 0.2}{0.4} = \frac{1}{3}.$$

2. Let W_j, R_j, B_j denote the event that a white ball, red ball, black ball, respectively, was selected on draw j.

(a) Using the law of total probability we find

$$\begin{aligned}
\mathbf{P}(B_2) &= \mathbf{P}(B_2 \mid W_1)\,\mathbf{P}(W_1) + \mathbf{P}(B_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(B_2 \mid B_1)\,\mathbf{P}(B_1) \\
&= \frac{b}{w+r+b+d} \cdot \frac{w}{w+r+b} + \frac{b}{w+r+b+d} \cdot \frac{r}{w+r+b} + \frac{b+d}{w+r+b+d} \cdot \frac{b}{w+r+b} \\
&= \frac{1}{w+r+b+d} \cdot \frac{1}{w+r+b}\,[bw + br + (b+d)b] \\
&= \frac{1}{w+r+b+d} \cdot \frac{1}{w+r+b} \cdot b\,[w+r+b+d] \\
&= \frac{b}{w+r+b}.
\end{aligned}$$

Notice that the probability that the second ball is black is the *same* as the probability that the first ball is black; that is, $\mathbf{P}(B_1) = \mathbf{P}(B_2) = b/(w+r+b)$. However, the events B_1 and B_2 are *not* independent. This is because

$$\mathbf{P}(B_1 \cap B_2) = \mathbf{P}(B_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{b+d}{w+r+b+d} \cdot \frac{b}{w+r+b+d}$$

which does not equal $\mathbf{P}(B_1)\,\mathbf{P}(B_2)$.

(b) Using Bayes' rule we find

$$\mathbf{P}(R_1 \mid B_2) = \frac{\mathbf{P}(B_2 \mid R_1)\,\mathbf{P}(R_1)}{\mathbf{P}(B_2)} = \frac{\frac{b}{w+r+b+d} \cdot \frac{r}{w+r+b}}{\frac{b}{w+r+b}} = \frac{r}{w+r+b+d}.$$

3. Notice that the fact that there are 85 students in the class is irrelevant. The average course grade is simply the weighted average of the assignment grades, midterm grades, and final exam grades, namely

$$\text{average course grade} = 93\% \cdot \frac{14}{100} + 75\% \cdot \frac{36}{100} + 80\% \cdot \frac{50}{100} = 80.02\%.$$

4. On any given roll, there are three things that can happen. Let A be the event $A = \{\text{roll a } 7\}$, let B be the event $B = \{\text{roll either a 6 or an 8}\}$, and let C be the event $C = \{\text{roll a something either than a 6, a 7, or an 8}\}$, so that

$$\mathbf{P}(A) = \frac{6}{36}, \quad \mathbf{P}(B) = \frac{10}{36}, \quad \mathbf{P}(C) = \frac{6}{36}.$$

Now, in order to roll a 7 before either a 6 or 8, you have to roll (i) a 7 on the first roll, or (ii) something either than a 6, 7, or 8 on the first roll, then a 7 on the second roll, or (iii) something either than a 6, 7, or 8 on each of the first two rolls, then a 7 on the third roll, etc. That is,

$$\begin{aligned}
\mathbf{P}(\text{roll a 7 before either a 6 or an 8}) &= \mathbf{P}(A \text{ or } CA \text{ or } CCA \text{ or } CCCA \text{ or } CCCCA \text{ or } \cdots) \\
&= \mathbf{P}(A) + \mathbf{P}(C)\,\mathbf{P}(A) + \mathbf{P}(C)^2\,\mathbf{P}(A) + \mathbf{P}(C)^3\,\mathbf{P}(A) + \cdots \\
&= \mathbf{P}(A)\left[1 + \mathbf{P}(C) + \mathbf{P}(C)^2 + \mathbf{P}(C)^3 + \cdots\right] \\
&= \mathbf{P}(A) \cdot \frac{1}{1 - \mathbf{P}(C)} \\
&= \frac{6}{36} \cdot \frac{1}{1 - \frac{20}{36}} \\
&= \frac{3}{8}.
\end{aligned}$$

Hence, we also find

$$\mathbf{P}(\text{roll either a 6 or an 8 before a 7}) = 1 - \mathbf{P}(\text{roll a 7 before either a 6 or an 8}) = \frac{5}{8}.$$

Therefore,

$$\mathbf{P}(X = -12) = \mathbf{P}(\text{roll a 7 before either a 6 or an 8}) = \frac{3}{8},$$

$$\mathbf{P}(X = 7) = \mathbf{P}(\text{roll either a 6 or an 8 before a 7}) = \frac{5}{8},$$

and

$$\mathbb{E}(X) = \sum_k k\mathbf{P}(x = k) = (-12)\mathbf{P}(X = -12) + (7)\mathbf{P}(X = 7) = -12 \cdot \frac{3}{8} + 7 \cdot \frac{5}{8} = -\frac{1}{8}.$$

5. If X is a continuous random variable, then

$$\mathbf{P}(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x.$$

We know from single-variable calculus that the value of a definite integral can be approximated by a Riemann sum. In particular, if a and b are close, then the approximation is generally quite good. That is,

$$\int_a^{a+\varepsilon} f(x)\,\mathrm{d}x \approx f(a)\cdot\varepsilon.$$

In this example, since

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

we find

(a) $\mathbf{P}(0 \le X \le 0.0001) \approx f(0)\cdot 0.0001 = \frac{1}{\sqrt{2\pi}}\cdot 0.0001 \doteq 0.00003989423$, and

(b) $\mathbf{P}(1 \le X \le 1.0001) \approx f(1)\cdot 0.0001 = \frac{1}{\sqrt{2\pi}}e^{-1/2}\cdot 0.0001 \doteq 0.00002419707$.

Note: If you used a midpoint Riemann sum instead, you would find

(a) $\mathbf{P}(0 \le X \le 0.0001) \approx f(0.00005)\cdot 0.0001 \doteq 0.00003989423$, and

(b) $\mathbf{P}(1 \le X \le 1.0001) \approx f(1.00005)\cdot 0.0001 \doteq 0.00002419586$.

6. (a) Observe that the expected distance that such a randomly selected tire lasts for is

$$\mathbb{E}(10000X) = 10000\cdot\mathbb{E}(X).$$

Since

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x = \int_1^2 x\cdot\frac{2}{x^2}\,\mathrm{d}x = 2\log x\Big|_1^2 = 2\log 2 - 2\log 1 = \log 4,$$

we conclude that the expected distance that such a randomly selected tire lasts for is

$$10000\log 4 \doteq 13862.94.$$

(b) Observe that the probability that a randomly selected tire lasts for between 10000 and 12500 kilometres given that it lasts for less than 15000 kilometres is equal to $\mathbf{P}(1 < X < 1.25\,|\,X < 1.5)$. By definition of conditional probability

$$\mathbf{P}(1 < X < 1.25\,|\,X < 1.5) = \frac{\mathbf{P}(1 < X < 1.25,\, X < 1.5)}{\mathbf{P}(X < 1.5)} = \frac{\mathbf{P}(1 < X < 1.25)}{\mathbf{P}(X < 1.5)}$$

using the fact that $\{1 < X < 1.25 \text{ and } X < 1.5\}$ if and only if $\{1 < X < 1.25\}$. We now find

$$\mathbf{P}(1 < X < 1.25) = \int_1^{1.25} f(x)\,\mathrm{d}x = \int_1^{1.25} 2x^{-2}\,\mathrm{d}x = -2x^{-1}\Big|_1^{1.25} = \frac{2}{5}$$

and

$$\mathbf{P}(X < 1.5) = \int_{-\infty}^{1.5} f(x)\,\mathrm{d}x = \int_1^{1.5} 2x^{-2}\,\mathrm{d}x = -2x^{-1}\Big|_1^{1.5} = \frac{2}{3},$$

so that the required probability is

$$\mathbf{P}(1 < X < 1.25\,|\,X < 1.5) = \frac{2/5}{2/3} = \frac{3}{5}.$$

7. (a) By the law of total probability, if we condition on the value of X, then

$$\mathbf{P}(Y > X) = \int_{-\infty}^{\infty} \mathbf{P}(Y > X \mid X = x) \, f_X(x) \, \mathrm{d}x.$$

Since $Y \sim \text{Exp}(\lambda_2)$, we find that if $x > 0$, then

$$\mathbf{P}(Y > X \mid X = x) = \mathbf{P}(Y > x) = \int_x^{\infty} \lambda_2 e^{-\lambda_2 y} \, \mathrm{d}y = e^{-\lambda_2 x}.$$

Therefore, since $X \sim \text{Exp}(\lambda_1)$,

$$\begin{aligned}
\mathbf{P}(Y > X) = \int_0^{\infty} \mathbf{P}(Y > X \mid X = x) \, f_X(x) \, \mathrm{d}x &= \int_0^{\infty} e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x} \, \mathrm{d}x \\
&= \lambda_1 \int_0^{\infty} e^{-(\lambda_1 + \lambda_2)x} \, \mathrm{d}x \\
&= -\frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)x} \Big|_0^{\infty} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

(b) To be updated later.

(c) By the law of total probability, if we condition on the value of X, then

$$\mathbf{P}(Y > X) = \int_{-\infty}^{\infty} \mathbf{P}(Y > X \mid X = x) \, f_X(x) \, \mathrm{d}x.$$

Now we need to be careful. We know that $Y \sim \text{Unif}(0, 1)$ so that the value $Y = y$ can only be between 0 and 1. This means that if we are given the value $X = x$ and $X = x$ is less than 0, then $X < Y$ with certainty. Similarly, if we are given the value $X = x$ and $X = x$ is greater than 1, then $X > Y$ with certainty. However, if we are given the value of $X = x$ and $X = x$ is between 0 and 1, then

$$\mathbf{P}(Y > X \mid X = x) = \mathbf{P}(Y > x) = \int_x^1 1 \, \mathrm{d}x = (1 - x) \;\text{ provided that }\; 0 < x < 1$$

In summary,

$$\mathbf{P}(Y > X \mid X = x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathbf{P}(Y > X \mid X = x) \, f_X(x) \, \mathrm{d}x &= \int_{-\infty}^{0} 1 \cdot f_X(x) \, \mathrm{d}x + \int_0^1 (1 - x) f_X(x) \, \mathrm{d}x + \int_1^{\infty} 0 \cdot f_X(x) \, \mathrm{d}x \\
&= \int_{-\infty}^{0} f_X(x) \, \mathrm{d}x + \int_0^1 f_X(x) \, \mathrm{d}x - \int_0^1 x f_X(x) \, \mathrm{d}x \\
&= \int_{-\infty}^{1} f_X(x) \, \mathrm{d}x - \int_0^1 x f_X(x) \, \mathrm{d}x.
\end{aligned}$$

Since $X \sim \mathcal{N}(0,1)$, we know that the value

$$\int_{-\infty}^{1} f_X(x)\,\mathrm{d}x = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \Phi(1)$$

can be computed from a table of normal probabilities. Doing so, we find $\Phi(1) \doteq 0.8413$. In order to compute

$$\int_{0}^{1} x f_X(x)\,\mathrm{d}x = \int_{0}^{1} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{0}^{1} x e^{-x^2/2}\,\mathrm{d}x$$

we make the substitution $u = x^2/2$, $\mathrm{d}u = x\,\mathrm{d}x$, so that

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{1} x e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{0}^{1/2} e^{-u}\,\mathrm{d}u = \frac{1}{\sqrt{2\pi}} \left[1 - e^{-1/2}\right] \doteq 0.1570.$$

Hence, we conclude

$$\mathbf{P}\left(Y > X\right) \doteq 0.8413 - 0.1570 = 0.6843.$$

8. If X_1 and X_2 are iid with common density

$$f(x) = xe^{-x^2/2}$$

for $x \geq 0$, then for $x \geq 0$ we have

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{0}^{x} te^{-t^2/2}\,\mathrm{d}t = 1 - e^{-x^2/2}.$$

Hence, the common distribution function of X_1 and X_2 is

$$F(x) = \begin{cases} 1 - e^{-x^2/2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(a) If $Y = \min\{X_1, X_2\}$, then the density function of Y is

$$f_Y(y) = 2f(y)[1 - F(y)] = 2 \cdot ye^{-y^2/2} \cdot [1 - (1 - e^{-y^2/2})] = 2 \cdot ye^{-y^2/2} \cdot e^{-y^2/2} = 2ye^{-y^2}$$

provided that $y \geq 0$.

(b) If $Z = Y^2$, then the distribution function of Z is

$$F_Z(z) = \mathbf{P}\left(Z \leq z\right) = \mathbf{P}\left(Y^2 \leq z\right) = \mathbf{P}\left(Y \leq \sqrt{z}\right) = \int_{-\infty}^{\sqrt{z}} f_Y(y)\,\mathrm{d}y = \int_{0}^{\sqrt{z}} 2ye^{-y^2}\,\mathrm{d}y = -e^{-y^2}\Big|_{0}^{\sqrt{z}}$$
$$= 1 - e^{-z}$$

for $z \geq 0$ and $F(z) = 0$ for $z < 0$. Hence, the density function of Z is $f_Z(z) = e^{-z}$ for $z \geq 0$. Note that $Z \sim \text{Exp}(1)$.

(c) There are two ways to compute $\mathbb{E}(Y^2)$. The first is

$$\mathbb{E}(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,\mathrm{d}y = \int_{0}^{\infty} y^2 \cdot 2ye^{-y^2}\,\mathrm{d}y = \int_{0}^{\infty} ue^{-u}\,\mathrm{d}u = \left[-ue^{-u} - e^{-u}\right]_0^\infty = 1.$$

The second is to use the fact that $Z = Y^2$; that is,

$$\mathbb{E}(Y^2) = \mathbb{E}(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,\mathrm{d}z = \int_{0}^{\infty} ze^{-z}\,\mathrm{d}z = \left[-ze^{-z} - e^{-z}\right]_0^\infty = 1.$$