Math/Stat 251 Fall 2015
Solutions to Assignment #5

1.

(a) Since
$$\int_1^\infty x^{-2}\,\mathrm{d}x = -x^{-1}\Big|_1^\infty = 0-(-1) = 1$$
we see that taking $c=1$ makes f a legitimate probability density.

(b) Since
$$\int_1^\infty x^{-1}\,\mathrm{d}x = \ln|x|\Big|_1^\infty = \infty - 0 = \infty$$
we see that there is no such c that makes f a legitimate probability density.

(c) Using integration-by-parts twice (see Prerequisite Review Handout) gives
$$\int x^2e^{-x}\,\mathrm{d}x = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}.$$

Therefore, since
$$\int_{-1}^1 x^2e^{-x}\,\mathrm{d}x = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\Big|_{-1}^1 = [-e^{-1}-2e^{-1}-2e^{-1}]-[-e^1+2e^1-2e^1] = e-5e^{-1},$$
we see that taking $c=(e-5e^{-1})^{-1} = e/(e^2-5)$ makes f a legitimate probability density.

(d) As in **(c)**, using integration-by-parts twice gives
$$\int_0^\infty x^2e^{-x}\,\mathrm{d}x = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\Big|_0^\infty = 0-(-2) = 2$$
and so we see that taking $c=1/2$ makes f a legitimate probability density.

(e) Since
$$\int_{-\infty}^0 xe^x\,\mathrm{d}x = [xe^x - e^x]\Big|_{-\infty}^0 = (0-1)-(0-0) = -1$$
by see that taking $c=-1$ makes f a legitimate probability density. Note that $xe^x \le 0$ for $x \le 0$. This means that if we multiply xe^x by a negative number it will always be non-negative. Hence, $f(x) = -xe^x$ for $x \le 0$ is a non-negative function that integrates to 1.

(f) Note that the function xe^x assumes both positive and negative values when $-1 \le x \le 1$. This means that there is no single value of c that could be multiplied by xe^x to make it strictly non-negative. Hence, there is no possible value of c that makes f a legitimate probability density.

2.

(a) We find
$$\mathbf{P}\{0<X<2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_0^2 7e^{-7x}\,\mathrm{d}x = -e^{-7x}\Big|_0^2 = 1-e^{-14}.$$

(b) We find

$$\mathbf{P}\{0 < X < 2\} = \int_0^2 f(x)\,\mathrm{d}x = \int_0^2 xe^{-x}\,\mathrm{d}x = \left[-xe^{-x} - e^{-x}\right]\Big|_0^2 = 1 - 3e^{-2}.$$

3. **(a)** Notice that we must necessarily have $0 < a < 4$. Since

$$\mathbf{P}\{X \le a\} = \int_0^a \frac{1}{8}x\,\mathrm{d}x = \frac{x^2}{16}\Big|_0^a = \frac{a^2}{16},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$\frac{a^2}{16} = \frac{1}{2}$$

implying that $a^2 = 8$. The restriction that $a > 0$ implies that the unique value of a such that $\mathbf{P}\{X \le a\} = 1/2$ is $a = \sqrt{8}$.

(b) As in **(a)**, we must necessarily have $0 < a < 4$. Since

$$\mathbf{P}\{X \ge a\} = \int_a^4 \frac{1}{8}x\,\mathrm{d}x = \frac{x^2}{16}\Big|_a^4 = 1 - \frac{a^2}{16},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$1 - \frac{a^2}{16} = \frac{1}{4}$$

implying that $a^2 = 12$. The restriction that $a > 0$ implies that the unique value of a such that $\mathbf{P}\{X \ge a\} = 1/4$ is $a = \sqrt{12}$.

4. Recall that the distribution function F of a random variable is defined as

$$F(x) = \mathbf{P}\{X \le x\}.$$

Since X is a continuous random variable, we know that $\mathbf{P}\{X = x\} = 0$ for any $x \in \mathbb{R}$ so that

$$\mathbf{P}\{X < x\} = \mathbf{P}\{X \le x\} = F(x).$$

(a) We find

$$\mathbf{P}\{X \le 1\} = F(1) = \frac{1}{8}(1)^3 = \frac{1}{8}.$$

(b) We find

$$\mathbf{P}\{0.5 \le X \le 1.5\} = \mathbf{P}\{X \le 1.5\} - \mathbf{P}\{X < 0.5\} = F(1.5) - F(0.5) = \frac{1}{8}\left[(1.5)^3 - (0.5)^3\right] = \frac{13}{32}.$$

(c) Notice that we must necessarily have $0 < a < 2$. Since

$$\mathbf{P}\{X \le a\} = F(a) = \frac{a^3}{8},$$

we find that in order for $\mathbf{P}\{X \le a\} = 1/2$ we must have

$$\frac{a^3}{8} = \frac{1}{2}$$

implying that $a^3 = 4$. Thus, the unique value of a such that $\mathbf{P}\{X \le a\} = 1/2$ is $a = 4^{1/3} = \sqrt[3]{4}$.

(d) As in **(c)**, we must necessarily have $0 < a < 2$. Since

$$\mathbf{P}\{X \ge a\} = 1 - \mathbf{P}\{X < a\} = 1 - F(a) = 1 - \frac{a^3}{8},$$

we find that in order for $\mathbf{P}\{X \ge a\} = 1/4$ we must have

$$1 - \frac{a^3}{8} = \frac{1}{4}$$

implying that $a^3 = 6$. Thus, the unique value of a such that $\mathbf{P}\{X \ge a\} = 1/4$ is $a = 6^{1/3} = \sqrt[3]{6}$.

5. Let

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t$$

denote the distribution function of a normally distributed random variable. Hence, we find the following.

(a) $\mathbf{P}\{X > 1\} = 1 - \mathbf{P}\{X \le 1\} = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) $\mathbf{P}\{X < 1\} = \mathbf{P}\{X \le 1\} = \Phi(1) = 0.8413$.

(c) $\mathbf{P}\{X \le 1\} = \Phi(1) = 0.8413$.

(d) $\mathbf{P}\{-1 \le X \le 1\} = \mathbf{P}\{X \le 1\} - \mathbf{P}\{X < -1\} = \mathbf{P}\{X \le 1\} - \mathbf{P}\{X \le -1\} = \Phi(1) - \Phi(-1) = (0.8413) - (1 - 0.8413) = 0.8413 - 0.1587 = 0.6826$.

(e) $\mathbf{P}\{X \le 2\} = \Phi(2) = 0.9772$.

(f) $\mathbf{P}\{X \ge -2\} = \mathbf{P}\{X \le 2\} = \Phi(2) = 0.9772$.

(g) $\mathbf{P}\{-2 \le X < 3\} = \mathbf{P}\{X < 3\} - \mathbf{P}\{X < -2\} = \mathbf{P}\{X \le 3\} - \mathbf{P}\{X \le -2\} = \Phi(3) - \Phi(-2) = (0.9987) - (1 - 0.9772) = 0.9987 - 0.0228 = 0.9759$.

(h) $\mathbf{P}\{-1 \le X \le 3\} = \mathbf{P}\{X < 3\} - \mathbf{P}\{X < -1\} = \mathbf{P}\{X \le 3\} - \mathbf{P}\{X \le -1\} = \Phi(3) - \Phi(-1) = 0.9987 - 0.1587 = 0.8400$.

6. **(a)** Let A_j, $j = 1, 2, 3$, denote the event that the lifetime of the jth TV lasts for at least two years. Therefore, since the TVs are selected at random and the TV lifetimes are independent, we find

$$\mathbf{P}\{A_1\} = \mathbf{P}\{A_2\} = \mathbf{P}\{A_3\} = \mathbf{P}\{X \ge 2\} = \int_2^\infty f(x)\,\mathrm{d}x = \int_2^\infty \frac{1}{2} e^{-x/2}\,\mathrm{d}x = -e^{-x/2}\Big|_2^\infty = \frac{1}{e}.$$

Hence, the probability that all three TVs last for at least two years is

$$\mathbf{P}\{A_1 \cap A_2 \cap A_3\} = \mathbf{P}\{A_1\}\,\mathbf{P}\{A_2\}\,\mathbf{P}\{A_3\} = \left(\frac{1}{e}\right)^3 = e^{-3}.$$

(b) Let B_j, $j = 1, 2, 3$, denote the event that the lifetime of the jth TV lasts for less than one year. Therefore, since the TVs are selected at random and the TV lifetimes are independent, we find

$$\mathbf{P}\{B_1\} = \mathbf{P}\{B_2\} = \mathbf{P}\{B_3\} = \mathbf{P}\{X < 1\} = \int_{-\infty}^{1} f(x)\,\mathrm{d}x = \int_0^1 \frac{1}{2}e^{-x/2}\,\mathrm{d}x = -e^{-x/2}\Big|_0^1 = 1 - e^{-1/2}.$$

Hence, the probability that exactly one TV last for less than one year

$$\begin{aligned}
&\mathbf{P}\{\text{exactly one TV lasts for less than one year}\}\\
&\quad = \mathbf{P}\{B_1 \cap B_2^c \cap B_3^c \text{ or } B_1^c \cap B_2 \cap B_3^c \text{ or } B_1^c \cap B_2^c \cap B_3\}\\
&\quad = \mathbf{P}\{B_1 \cap B_2^c \cap B_3^c\} + \mathbf{P}\{B_1^c \cap B_2 \cap B_3^c\} + \mathbf{P}\{B_1^c \cap B_2^c \cap B_3\}\\
&\quad = \mathbf{P}\{B_1\}\,\mathbf{P}\{B_2^c\}\,\mathbf{P}\{B_3^c\} + \mathbf{P}\{B_1^c\}\,\mathbf{P}\{B_2\}\,\mathbf{P}\{B_3^c\} + \mathbf{P}\{B_1^c\}\,\mathbf{P}\{B_2^c\}\,\mathbf{P}\{B_3\}\\
&\quad = (1 - e^{-1/2})(e^{-1/2})^2 + (1 - e^{-1/2})(e^{-1/2})^2 + (1 - e^{-1/2})(e^{-1/2})^2\\
&\quad = 3e^{-1}(1 - e^{-1/2}).
\end{aligned}$$

7. (You may want to draw a tree diagram to help interpret the solution.) Let A be the event that a randomly selected Toyota vehicle is recalled. Let B_1 be the event that a randomly selected Toyota vehicle is a car, let B_2 be the event that a randomly selected Toyota vehicle is a truck, and let B_3 be the event that a randomly selected Toyota vehicle is a van. We are told that $\mathbf{P}\{B_1\} = 0.65$, $\mathbf{P}\{B_2\} = 0.20$, and $\mathbf{P}\{B_3\} = 0.15$. We are also told that $\mathbf{P}\{A \,|\, B_1\} = 0.10$, $\mathbf{P}\{A \,|\, B_2\} = 0.08$, and $\mathbf{P}\{A \,|\, B_3\} = 0.12$. We want to determine $\mathbf{P}\{B_2 \,|\, A\}$. Using Bayes' Rule we find

$$\begin{aligned}
\mathbf{P}\{B_2 \,|\, A\} = \frac{\mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\}}{\mathbf{P}\{A\}} &= \frac{\mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\}}{\mathbf{P}\{A \,|\, B_1\}\,\mathbf{P}\{B_1\} + \mathbf{P}\{A \,|\, B_2\}\,\mathbf{P}\{B_2\} + \mathbf{P}\{A \,|\, B_3\}\,\mathbf{P}\{B_3\}}\\
&= \frac{(0.08)(0.20)}{(0.10)(0.65) + (0.08)(0.20) + (0.12)(0.15)}\\
&= \frac{16}{99} \doteq 0.161616.
\end{aligned}$$