Math 026L.04 Spring 2002
Test #3 Solutions

1. Let $f(x) = e^{-ax^2}$.

(a) LHS= $\frac{1}{2}f(0) + \frac{1}{2}f(1/2) = \frac{1}{2}(1 + e^{-a/4})$

(b) RHS= $\frac{1}{2}f(1/2) + \frac{1}{2}f(1) = \frac{1}{2}(e^{-a/4} + e^{-a})$

(c) LHS= $\dfrac{1}{100}\sum_{k=0}^{99} f(k/100) = \dfrac{1}{100}\sum_{k=0}^{99} e^{-k^2/20000} = .857587.$

(d) RHS= $\dfrac{1}{100}\sum_{k=1}^{100} f(k/100) = \dfrac{1}{100}\sum_{k=1}^{100} e^{-k^2/20000} = .853652.$

(e) Since the $f(x)$ is strictly decreasing on the interval $[0,1]$, we have

$$.853652 < \int_0^1 e^{-\frac{x^2}{2}}\, dx < .857587.$$

2.

(a) $\displaystyle\int x^2 + \frac{1}{\sqrt{x}}\, dx = \frac{1}{3}x^3 + 2\sqrt{x} + C$

(b) $\displaystyle\int x(\ln x)^2\, dx = \frac{x^2}{2}(\ln x)^2 - \int x\ln x\, dx = \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx\right)$

$\displaystyle= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$

(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\, dx = -\frac{1}{4}\sqrt{1-4x^2} + C$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\, dx = \frac{1}{2}\arcsin(2x) + C$

3.

(a) FTC Part I: $\displaystyle\int_a^b f'(t)\, dt = f(b) - f(a).$

(b) FTC Part II: $\displaystyle\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$

4.

(a) $\displaystyle F(6) = \int_0^6 f(t)\, dt = 1.5.$

(b) $\displaystyle F'(x) = \frac{d}{dx}\int_0^x f(t)\, dt = f(x).$ Thus, $F'(1) = f(1) = -1.$

(c) $g(4) = \int_0^4 f'(t)\,dt = f(4) - f(0) = 1 - (-1) = 2.$

(d) $g'(x) = \dfrac{d}{dx}\int_0^x f'(t)\,dt = f'(x)$. Thus, $g'(2) = f'(2) = 1.$

5.

(a) $\int_1^4 2f(x) - 3g(x) + 4\,dx = 2\int_1^4 f(x)\,dx - 3\int_1^4 g(x)\,dx + \int_1^4 4\,dx = 2\cdot 5 - 3\cdot 9 + 4(4-1) = -5.$

(b) $\int_7^1 f(x)\,dx = -\int_1^7 f(x) = -\left(\int_1^4 f(x)\,dx + \int_4^7 f(x)\,dx\right) = -(5 + -3) = -2.$

(c) $\dfrac{1}{7-1}\int_1^7 g(x)\,dx = \dfrac{2}{6}\int_1^4 g(x)\,dx = \dfrac{1}{3}\cdot 9 = 3$ since $g(x)$ is even.

6.

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx.$

Since $\int \frac{1}{1+x^2}\,dx = \arctan x + C$, we have

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}[(\arctan 0 + C) - (\arctan a + C)] = -\lim_{a\to-\infty}\arctan a = \frac{\pi}{2}.$$

Also,

$$\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}[(\arctan b + C) - (\arctan 0 + C)] = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}.$$

Combining gives $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$ so that $K = \frac{1}{\pi}$.

7.

(a) Draw two straight line segments: one connecting the points $(0, 0^{3/2})$ and $(1, 1^{3/2})$, the other connecting the points $(1, 1^{3/2})$ and $(2, 2^{3/2})$. Then,

$$\ell \approx \sqrt{(1-0)^2 + (1^{3/2} - 0^{3/2})^2} + \sqrt{(2-1)^2 + (2^{3/2} - 1^{3/2})^2} \approx 3.498.$$

(b) $\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x \quad\approx\quad \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

(c) If $f(x) = x^{3/2}$, then $f'(x) = \frac{3}{2}\sqrt{x}$, so that

$$\begin{aligned}\ell &= \int_0^2 \sqrt{1 + \left[\frac{3}{2}\sqrt{x}\right]^2}\,dx = \int_0^2 \sqrt{1 + \frac{9}{4}x}\,dx = \left[\frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2} + C\right]_0^2 \\ &= \frac{8}{27}\left(\left(1 + \frac{18}{8}\right)^{3/2} - 1\right) \\ &= \frac{(13)^{3/2} - 8}{27} \approx 1.44\end{aligned}$$