Math 026L.04 Spring 2002
Test #2

Name: ___________________________________

Read all of the following information before starting the test:

- Be sure that this test has **11** numbered pages.
- There are **7** problems on this test, plus some bonus problems at the end, worth a total of **100** points.
- The last page is for your scrap work and may be detached from the test booklet.
- Calculators are permitted, but no other aids are allowed. When you do use your calculator, sketch all relevant graphs and write down all relevant mathematics.
- Show all work neatly and in order, and clearly indicate your final answers.
- Answers must be justified whenever possible in order to earn full credit. ***No credit will be given for unsupported answers, even if your final answer is correct.***
- Please keep your written answers succinct. Points will be deducted for incoherent, incorrect and/or irrelevant statements.
- Good luck!

Problem	1	2	3	4	5	6	7	B	S	Total
Score										

1. (*16 points*) Solve the following initial value problems.

(a) $2\dfrac{dy}{dx} + x\dfrac{dy}{dx} = 1,\ y(0) = 0$

(b) $\dfrac{y^2}{\cos x} + \dfrac{1}{\cos x} - \cos x\dfrac{dy}{dx} = 0,\ y(0) = 0$

2. (*14 points*) A tank contains 1000 L of water with 150 kg of sugar dissolved in it. Pure water enters the tank at a rate of 20 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

(a) How much sugar is in the tank after 20 minutes?

(b) After a very long time, what is the concentration of sugar in the tank? Is this surprising? Why or why not?

(continued)

Suppose instead of pure water, water containing 0.5 kg/L of sugar enters the tank.

(c) In this case, how much sugar would there be in the tank after 20 minutes?

(d) After a very long time, what is the concentration of sugar in the tank? Is this surprising? Why or why not?

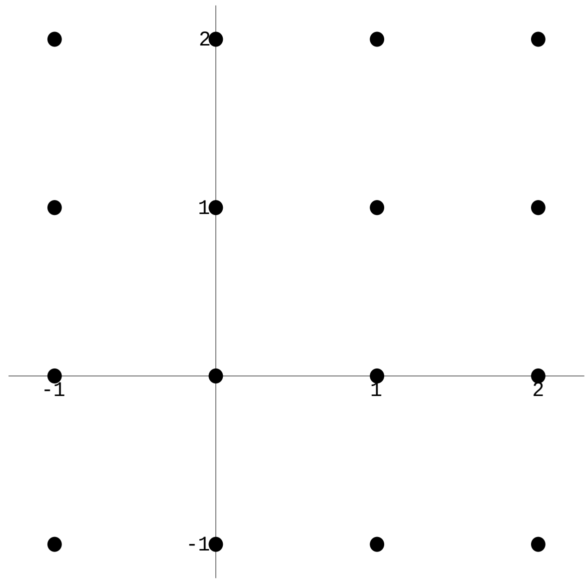
3. (*14 points*) The population of the island of Åland in the Gulf of Bothnia on January 1, 1990, was 1400. Census data taken there indicate that if there is no immigration or emmigration, the population grows at a rate proportional to itself, with constant of proportionality .04.

(a) Given these conditions, write and solve an initial value problem for the population function $P(t)$. What does your function predict that the population will be on January 1, 2002?

(b) Now suppose that in addition 30 people immigrate to the island every year. Find and solve a new initial value problem using this new assumption. What does this function predict the population will be on January 1, 2002?

4. (*16 points*) Consider the differential equation $\dfrac{dy}{dx} = x + y - 1$.

(a) Sketch the slope field for this differential equation using the grid below.



(b) On the slopefield your drew above, sketch the solution curve corresponding to the initial value $y(0) = 1$.

(c) Suppose that $y(0) = 1$. Use Euler's method with 4 steps to approximate $y(2)$.

5. (*12 points*) The passenger pigeon was present in the United States in vast numbers until late in the nineteenth century when it was heavily hunted for food and sport and its numbers were drastically reduced. Unfortunately, the passenger pigeon could only breed successfully when present in a large concentration. It was believed that the population of the passenger pigeon could be modeled by the differential equation

$$\frac{dP}{dt} = -P(P - 1000)(P - 2000).$$

(a) Does this differential equation have any equilibrium solutions? If so, identify each equilibrium solution as either stable or unstable. Be sure to justify your solution.

(b) Suppose that the initial population of passenger pigeons was 900. In the long run, does the population become extinct? Why or why not?

(c) Suppose that the initial population of passenger pigeons was 1100. In the long run, does the population become extinct? Why or why not?

6. (*16 points*)
One model for the spread of a disease among a population is the logistic growth model. In that model, the rate of spread is proportional to the product of the fraction of the population who have the disease and those who do not have the disease. One problem with this model is that, in the long run, the entire population becomes infected with the disease.

In this problem, we will outline a more sophisticated model for the spread of a disease.

Suppose that through vaccinations, there is a segment of the population that is not susceptible to the disease. Unfortunately, however, there remains a segment of the population that is susceptible to the disease. Let $S(t)$ be the number of susceptibles in the population at time t.

Suppose further that the disease is spread by *carriers* in the population. Let $C(t)$ denote the number of carriers in the population at time t.

In order to limit the spread of the disease, carriers are identified and removed from the population.

(a) Suppose that carriers are identified and removed at a rate β ($\beta > 0$) so that

$$\frac{dC}{dt} = -\beta C.$$

Determine $C(t)$ by solving the above equation assuming $C(0) = C_0$.

(continued)

(b) Also suppose that the disease spreads at a rate proportional to the product of S and C with constant of proportionality α ($\alpha > 0$) so that

$$\frac{dS}{dt} = \alpha SC.$$

Using your result from **(a)** determine $S(t)$ assuming that $S(0) = S_0$.

(c) Determine the proportion of the population that escapes the disease by finding the limiting value of S as $t \to \infty$.

7. (*12 points*)

(a) Write the following series in Σ-notation:

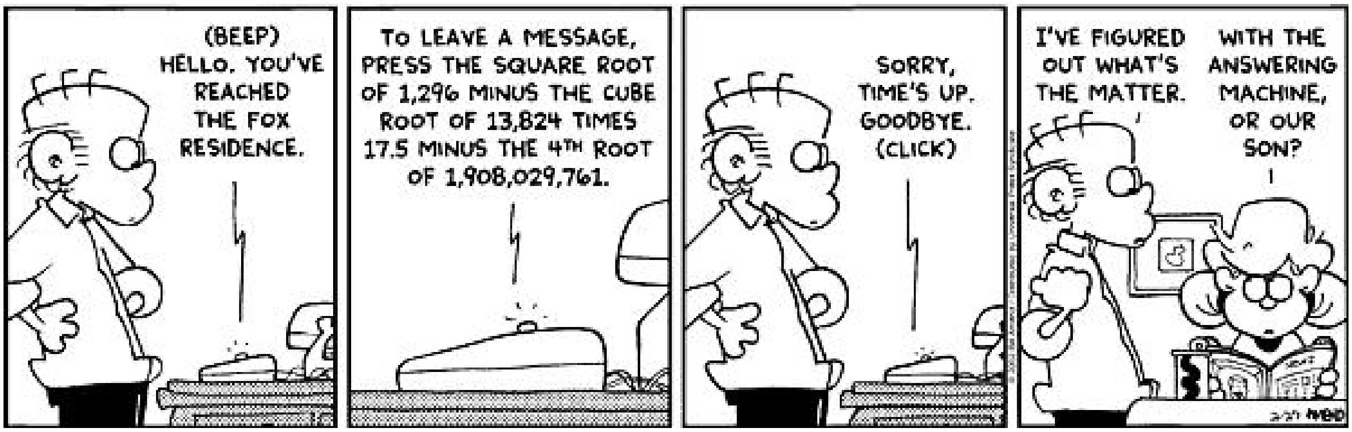
$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \cdots$$

(Hint: For each term, look for something multiplied by a power of two.)

(b) Approximate the value of this series accurate to three decimal places. How do you know that your approximation is accurate to three decimal places? Explain.

(c) What is $\ln 2$ accurate to three decimal places. What do you think this say about the series from **(a)**?

Bonus Question (*2 bonus points*)



What number should you press to leave a message for the Fox family?

Survey Question (*1 bonus point*)

What did you think of this test? Was it what you were expecting?

Scrap Page

(*You may carefully remove this page from the test booklet.*)