Math 26L.04 Spring 2002
Extra Probability Problem

**1.** Consider the following game. A player flips a fair coin. If the coins shows heads, then the player rolls two dice and wins $\$X$, where $X$ is the sum of the upmost faces on the two dice. However, if the coin shows tails, then the player rolls only one die and instead wins $\$X$, where $X$ is the upmost face on this single die.

**(a)** What are all possible values for $X$, the player's winnings?

The possible values for $X$ are: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12.

**(b)** Compute the probability mass density of $X$. Display your answer in a table.

DRAW A TREE DIAGRAM TO HELP YOU KEEP TRACK WHILE READING THIS SOLUTION.

With probability 1/2, the player flips a tails. If that happens then with probability 1/6, the player wins either 1, 2, 3, 4, 5, or 6.

With probability 1/2, the player flips a heads. If that happens then the player wins 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12, with probabilities

| Sum | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Probability | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

Now, we take into account whether one die or two was rolled. The only way to win \$1 is to flip a tails, and then roll a 1. Thus, $\mathbb{P}(X = 1) = \frac{1}{12} = \frac{6}{72}$.

The only way to win \$7, 8, 9, 10, 11, or 12 is to flip a heads, and then roll that sum. Thus, $\mathbb{P}(X = 7) = \frac{6}{72}$, $\mathbb{P}(X = 8) = \frac{5}{72}$, $\mathbb{P}(X = 9) = \frac{4}{72}$, $\mathbb{P}(X = 10) = \frac{3}{72}$, $\mathbb{P}(X = 11) = \frac{2}{72}$, $\mathbb{P}(X = 12) = \frac{1}{72}$.

However, it is possible to win \$2, 3, 4, 5, or 6, by rolling either one die or two. Thus, $\mathbb{P}(X = 2) = \frac{1}{12} + \frac{1}{72}$, $\mathbb{P}(X = 3) = \frac{1}{12} + \frac{2}{72}$, $\mathbb{P}(X = 4) = \frac{1}{12} + \frac{3}{72}$, $\mathbb{P}(X = 5) = \frac{1}{12} + \frac{4}{72}$, $\mathbb{P}(X = 6) = \frac{1}{12} + \frac{5}{72}$.

In summary,

| $X = k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(X = k)$ | $\frac{6}{72}$ | $\frac{7}{72}$ | $\frac{8}{72}$ | $\frac{9}{72}$ | $\frac{10}{72}$ | $\frac{11}{72}$ | $\frac{6}{72}$ | $\frac{5}{72}$ | $\frac{4}{72}$ | $\frac{3}{72}$ | $\frac{2}{72}$ | $\frac{1}{72}$ |

*(Continued)*

**(c)** Determine what a fair price to pay to play is by computing $\mathbb{E}(X)$.

$$\begin{aligned}
\mathbb{E}(X) &= 1\mathbb{P}(X=1) + 2\mathbb{P}(X=2) + \cdots + 11\mathbb{P}(X=11) + 12\mathbb{P}(X=12) \\
&= \sum_{k=1}^{12} k \cdot \mathbb{P}(X=k) \\
&= \frac{6+14+24+36+50+66+42+40+36+30+22+12}{72} \\
&= 5.25
\end{aligned}$$