Math 026L.04 Spring 2002

The Definite Integral

Suppose that $f(x)$ is a continuous, positive function defined on the interval $[a, b]$. We denote by

$$\int_a^b f(x)\, dx \tag{1}$$

the area under the curve $f(x)$, above the x-axis, and between $x = a$ and $x = b$. We call (1) the **definite integral of** $f(x)$ **from** a **to** b.

Properties of the Definite Integral

Fact: $\displaystyle\int_a^b 1\, dx = b - a$

Fact: If $a < b < c$, then $\displaystyle\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$.

Fact: $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Fact: $\displaystyle\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

Fact: $\displaystyle\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$

Exercise: Draw pictures of the associated areas to convince yourself that the above facts are true.

Riemann Sums

Only in rare cases is it possible to compute definite integrals exactly. Although the Fundamental Theorem of Calculus, which we will study shortly, will greatly expand the class of so-called integrable functions, the majority of definite integrals can only be approximated.

We construct Riemann sums to approximate (1). Suppose that we partition (subdivide) the interval $[a, b]$ into N equal subintervals each of length

$$\Delta x = \frac{b - a}{N}.$$

If we denote the partition points by

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_k = a + k\Delta x,\ \ldots,\ x_N = a + N\Delta x = b,$$

then the left hand sum (LHS) and right hand sum (RHS) are computed as follows.

- **LHS**: $\displaystyle\Delta x \sum_{k=0}^{N-1} f(x_k) = \Delta x \sum_{k=0}^{N-1} f(a + k\Delta x)$
- **RHS**: $\displaystyle\Delta x \sum_{k=1}^{N} f(x_k) = \Delta x \sum_{k=1}^{N} f(a + k\Delta x)$

Exercise: Make sure you understand these sums.

Example: Suppose $f(x) = 2x^3+4$. Compute the LHS and RHS to approximate $\int_{-1}^{1} f(x)\,dx$ using $N = 200$ subintervals.

Solution: We partition $[-1, 1]$ into 200 subintervals each of width $\Delta x = \frac{1 - -1}{200} = \frac{1}{100}$.

Thus, the partition points are -1, $-1 + \frac{1}{100}$, $-1 + \frac{2}{100}$, ..., $-1 + \frac{199}{100}$, 1.

Hence,

$$LHS = \Delta x \sum_{k=0}^{N-1} f(a + k\Delta x) = \frac{1}{100} \sum_{k=0}^{199} f\left(-1 + \frac{k}{100}\right) = \frac{1}{100} \sum_{k=0}^{199} \left[2\left(-1 + \frac{k}{100}\right)^3 + 4\right].$$

Similarly,

$$RHS = \Delta x \sum_{k=1}^{N} f(a + k\Delta x) = \frac{1}{100} \sum_{k=1}^{200} f\left(-1 + \frac{k}{100}\right) = \frac{1}{100} \sum_{k=1}^{200} \left[2\left(-1 + \frac{k}{100}\right)^3 + 4\right].$$

We can compute each of these on the TI-83 as follows:

- **LHS**: `sum(seq(1/100*(2*(-1+X/100)∧3 + 4), X, 0, 199, 1))`= 7.98
- **RHS**: `sum(seq(1/100*(2*(-1+X/100)∧3 + 4), X, 1, 200, 1))`= 8.02

Thus,

$$7.98 < \int_{-1}^{1} 2x^3 + 4\,dx < 8.02.$$

Note that sum is found as [2nd] [LIST] **MATH 5:sum** and seq is found as [2nd] [LIST] **OPS 5:seq**.

The Program `RIEMANN`

It is tedious to repeatedly enter such sums manually. The following simple program computes the LHS and RHS for $\int_a^b f(x)\,dx$ using N subintervals. You must enter $f(x)$ as `Y1`.

Either input this program directly into your TI-83, or copy it from someone using the TI Graph Link.

```
:Prompt A,B,N
:(B-A)/N -> D
:A -> X
:0 -> S
:For (I,0,N-1)
:S+Y1*D -> S
:X+D -> X
:End
:S -> L
:S+Y1*D -> S
:A -> X
:S-Y1*D -> R
:Disp "RIGHT SUM IS", R
:Disp "LEFT SUM IS", L
```

One final warning: Although this program will save you time on your homework and while you study, you will still be required to know how to set up Riemann sums on paper and how to compute them directly on your calculator without the aid of a program.