Mathematics 103.04 and 103.05 **Test Three**

I have neither given nor received aid in the completion of this test.
Signature:

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To get full credit you must show enough work to convince me that you know what you are doing!

1. 10 pts. Evaluate:

$$\int_0^1 \Big(\int_{x^2}^{x} \frac{2y}{x^2}\, dy\Big)\, dx.$$

Solution.

$$\int_0^1 \frac{y^2}{x^2}\Big|_{y=x^2}^{y=x} dx \;=\; \int_0^1 \frac{x^2 - x^4}{x^2}\, dx \;=\; \int_0^1 1 - x^2\, dx \;=\; \frac{2}{3}.$$

2. Let R be the region in $\mathbf{R}^2$ within the circle $x^2 + y^2 = 2$ and above the line $y = x$.

a. 10 pts. Express

$$I \;=\; \iint_R \sqrt{1 + x^2 + y^2}\, dxdy$$

as a sum of two iterated integrals in rectangular coordinates.

Solution. Draw a picture. Look at what happens when x is between $-\sqrt{2}$ and -1 and when x is between -1 and 1. Conclude that

$$\int_{-\sqrt{2}}^{-1}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \sqrt{1 + x^2 + y^2}\, dy\, dx \;+\; \int_{-1}^{1}\int_{x}^{\sqrt{2-x^2}} \sqrt{1 + x^2 + y^2}\, dy\, dx.$$

b. 10 pts. Express I as an iterated integral using polar coordinates.

Solution.

$$\int_{\pi/4}^{5\pi/4}\int_0^{\sqrt{2}} \sqrt{1 + r^2}\, r\, dr\, d\theta.$$

3. Let R be the part of the region in $\mathbf{R}^3$ bounded by $x^2 + y^2 + z^2 = 2$ which is above $z = \sqrt{x^2 + y^2}$.

a. 10 pts. Express

$$I \;=\; \iiint_R x^2 + y^2\, dxdydz$$

as an iterated integral in rectangular coordinates.

Solution. Note that the projection of R on the xy-plane is the circle of radius 1 with center at the origin. Thus

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x^2+y^2\,dz\,dy\,dx.$$

b. 10 pts. Express I as an iterated integral in cylindrical coordinates.

Solution.

$$\int_0^1\int_0^{2\pi}\int_r^{\sqrt{2-r^2-z^2}} r^2\,r\,dz\,d\theta\,dr.$$

c. 10 pts. Express I as an iterated integral in spherical coordinates.

Solution.

$$\int_0^{\sqrt{2}}\int_0^{\pi/4}\int_0^{2\pi}(\rho\sin\phi)^2\,\rho^2\sin\phi\,d\theta\,d\phi\,d\rho.$$

4. 10 pts. Let A be the surface area of the part of $z=\sqrt{x^2+y^2}$ below $z=2$. Express A as an iterated integral.

Solution. We may parameterize this surface by $\mathbf{r}(r,\theta)=(r\cos\theta, r\sin\theta, r)$ where $0<r<2$ and $0<\theta<2\pi$. We have

$$\mathbf{r}_r\ =\ (\cos\theta,\sin\theta,1),\quad \mathbf{r}_\theta\ =\ r(-\sin\theta,\cos\theta,0);$$

Since these vector are perpendicular the length of their cross product is the product of their lengths. Thus

$$|\mathbf{r}_r\times\mathbf{r}_\theta|\ =\ \sqrt{2}r.$$

so the answer is

$$\int_0^2\int_0^{2\pi}\sqrt{2}r\,d\theta\,dr.$$

5. 10 pts. Express the area of the part of the cylinder $x^2+y^2=1$ where $0<z<x^2$ as an iterated integral.

Solution. We may parameterize the cylinder $x^2+y^2=1$ by $\mathbf{r}(\theta,z)=(\cos\theta,\sin\theta,z)$ where $-\infty<r<\infty$ and $0<\theta<2\pi$. We have

$$\mathbf{r}_\theta\ =\ (-\sin\theta,\cos\theta,0),\quad \mathbf{r}_z\ =\ (0,0,1).$$

DSince these vector are perpendicular the length of their cross product is the product of their lengths. Thus

$$|\mathbf{r}_\theta\times\mathbf{r}_z|\ =\ 1.$$

Now we only want the area where $0<z<x^2=(r\cos\theta)^2$ so the answer is

$$\int_0^{2\pi}\int_0^{(r\cos\theta)^2}dz\,d\theta.$$

6. 10 pts. Let R be the region in $\mathbf{R}^2$ bounded by $x=1$, $x=2$, $xy=3$ and $xy=4$. Use the change of variables $u=x,\ v=xy$ to compute the area of R.

Solution. We have $x = u,\ y = v/u$. Letting $T(u,v) = (u, v/u)$ for (u,v) in $Q = (1,2) \times (3,4)$ we find that T carries Q in one-to-one fashion onto R. Moreover

$$J_T(u,v) \ = \ \mathbf{det} \begin{bmatrix} x_u & x_v \\ & \\ y_u & y_v \end{bmatrix} \ = \ \begin{bmatrix} 1 & 0 \\ & \\ -v/u^2 & 1/u \end{bmatrix}$$

the determinant of which is $1/u$. Thus

$$\text{Area}\, R \ = \ \iint_Q J_T(u,v)\, dudv \ = \ \int_1^2 \int_3^4 \frac{1}{u}\, dv\, du.$$

That's all folks!