Math 103.01 Summer 2001
Assignment #2
Due: Thursday, May 31, 2001

You must work through all problems on your own. You may consult any reference materials but do not discuss these problems with anyone else in the class. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.

1. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\left(\mathbf{r}(t) \cdot \mathbf{r}'(t)\right)$.

2. Suppose that $\mathbf{r}(t)$ is a vector-valued function with $\mathbf{r}'(t) \neq 0$. Suppose that $s = s(t)$ is its arc length function. Recall that the unit tangent vector is given by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, the curvature is $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$, the principal unit normal vector is $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, and the binormal vector is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

The following, known as the Frenet-Serret formulas, are of fundamental importance in differential geometry. Prove the first 2 identities and then use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce the third one.

- $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$
- $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$
- $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$

Note that τ is called the torsion of the curve.

3. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three vectors such that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and have the same length ℓ. Let $\mathbf{r}(t) = \mathbf{u}\cos t + \mathbf{v}\sin t + \mathbf{w}$.

a. Describe the motion of the tip of $\mathbf{r}(t)$ if the tail of $\mathbf{r}(t)$ is fixed at the origin. (That is, describe the parametric curve corresponding to $\mathbf{r}(t)$.)

b. Find the component functions of $\mathbf{r}(t)$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} - \mathbf{k}$, $\mathbf{w} = \mathbf{j} + \mathbf{k}$.

c. Compute $D_t(\mathbf{r}(t))$.

d. Compute $\displaystyle\int_0^{\pi} \mathbf{r}(t)\, dt$.

4. The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1.$$

5. Do the following limits exist? If so, compute the limit. If not, explain why.

a. $$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

b. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$