Math 171 Prelim #1 – Solutions

1. (a) Let X be the monthly income of a randomly chosen male Cornell graduate. Let Y be the monthly income of a randomly chosen female Cornell graduate. The total monthly income is then $X+Y$ and the difference is $X-Y$. (Note the difference is not $|X-Y|$, which is called the absolute difference. There is no way you may calculate the expectation of the absolute difference given the information. It is equivalent to define the difference as $Y-X$.)

(b) No extra assumption is needed for the following calculations. The mean of the total income is $E(X+Y) = E(X) + E(Y) = 5000 + 5000 = 10,000$, and the mean of the difference is $E(X-Y) = E(X) - E(Y) = 5000 - 5000 = 0$.

(c) We need to assume that X and Y are independent in the following calculations: $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y) = (2000)^2+(3000)^2 = 13,000,000$. Hence $\text{SD}(X+Y) = \sqrt{\text{Var}(X+Y)} = \sqrt{13,000,000} = 3,605$. Similarly, $\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y) = (2000)^2+(3000)^2 = 13,000,000$ so that $\text{SD}(X-Y) = \sqrt{\text{Var}(X-Y)} = \sqrt{13,000,000} = 3,605$.

2. (a) The two-engine ValueJet plane is not operational if no engines are functioning. Therefore, let X be the number of functioning engines so that X is binomial $p = 0.65$, $n = 2$. Hence,

$$P(X=0) = \frac{2!}{0!2!}(0.65)^0(0.35)^2 = 0.1225.$$

Thus, the probability the two-engine plane is operational is 0.8775.

(b) The four-engine ValueJet plane is not operational if none or only one engine is functioning. Hence, let Y be binomial with $p = 0.65$, $n = 4$. Hence,

$$P(Y=0)+P(Y=1) = \frac{4!}{0!4!}(0.65)^0(0.35)^4+\frac{4!}{1!3!}(0.65)^1(0.35)^3 = 0.0150+0.1115 = 0.1265.$$

Thus, the probability the two-engine plane is operational is 0.8735.

Note that the two-engine plane is safer to operate.

3. (a) $S = \{R, GH, GT, YHH, YHT, YTH, YTT\}$ where R denotes a roll of red, G a roll of green, and Y a roll of yellow, H a head on the toss of the coin, and T a tail.

(b) $P(A) = P(R) = 1/2$

(c) $P(B) = P(HH|Y) \cdot P(Y) = 1/4 \cdot 1/6 = 1/24$

(d) Since $A \cap B = \emptyset$, $P(A \cap B) = 0$. However, $P(A) \cdot P(B) = 1/48 \neq 0$ so that A and B are NOT independent.

4. **(a)** Let D be the event that the plant is dead when you return, and let W be the event that your neighbor watered the plant.

$$P(D) = P(D \cap W) + P(D \cap W^c) = P(D|W) \cdot P(W) + P(D|W^c) \cdot P(W^c)$$
$$= 0.15 * 0.90 + 0.80 * 0.10 = 0.215.$$

Hence the probability the plant is alive when you return is 0.785.

(b) We seek $P(W^c|D)$ which can be computed via

$$P(W^c|D) = \frac{P(D|W^c) \cdot P(W^c)}{P(D)} = \frac{0.80 * 0.10}{0.215} = 0.372.$$

5. Let X be normally distributed with mean 12 and standard deviation 3.

(a) $P(9 \leq X \leq 18) = P(-1 \leq Z \leq 2) = 0.9772 - 0.1587 = 0.8185$ where Z is normal mean 1 SD 0.

(b) If $0.20 = P(X < L) = P(Z < \frac{L-12}{3})$, then $(L - 12)/3 = -0.84$ so that $L = 9.48$.

(c) Let X_1 and X_2 be independent normal mean 12 SD 3. Then the total precipitation is $X_1 + X_2$ so that $E(X_1 + X_2) = E(X_1) + E(X_2) = 12 + 12 = 24$, and the variance of the total is $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 9 + 9 = 18$ since X_1 and X_2 are independent. Thus, $\mathrm{SD}(X + Y) = \sqrt{18} = 4.2426$.

6. Let X_1 be 1 or 0 depending on whether the first coin gives you a head or tail, and let X_2 be 1 or 0 depending on whether the second coin gives you a head or tail.

(a) The total number of heads is $X_1 + X_2$ so that the expectation of the total number of heads is

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 0.6 + 0.3 = 0.9.$$

(b) The variance of the total number of heads is

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 0.6 * 0.4 + 0.3 * 0.7 = 0.45.$$

(c) The probability of getting exactly one head is

$$0.6 * 0.7 + 0.4 * 0.3 = 0.54.$$

7. **(a)** Let $\hat{p}$ be the distribution of the sample proportion. The approximate distribution of $\hat{p}$ is normal with mean $p = 0.14$ and standard deviation $\sqrt{p(1-p)/n} = \sqrt{0.14 * 0.86/500} = 0.0155$.

(b) Using a normal probability calculation (CLT calculation) gives

$$P(\hat{p} \geq 0.15) = P\left(\frac{\hat{p} - 0.14}{0.0155} \geq \frac{0.15 - 0.14}{0.0155}\right) = P(Z \geq 0.645) = 1 - 0.74 = 0.26$$

where Z is normal mean 0 SD 1, so that our sample is 26% likely to contain at least 15% who own Harleys. It is up to you to judge if this constitutes "likely."

Note: This is problem 18.4 from Moore.