Math 111 Prelim #2 Solutions - July 21, 2003

1.

(a) $f'(x) = e^x \cos(e^x - 1) - \dfrac{x}{\sqrt{x^2+1}}$

(b) $f(x) = \dfrac{x}{2}\ln(x)$, so $f'(x) = \frac{1}{2}\ln x + \frac{1}{2}$.

(c) $f'(x) = 0$ since $e^\pi - \pi^e$ is a constant.

(d) $f'(x) = \dfrac{2^x \sec^2 x - 2^x \ln 2 \tan x}{2^{2x}} = \dfrac{\sec^2 x - \ln 2 \tan x}{2^x}$

2.

(a) We can use the chain rule to compute the required derivatives. Thus,

$$\frac{dx}{dt} = 4 \cdot 3\cos^2 t \cdot (-\sin t) = -12 \sin t \cos^2 t$$

and

$$\frac{dy}{dt} = 4 \cdot 3 \sin^2 t \cdot \cos t = 12 \cos t \sin^2 t.$$

(b) We can also use the chain rule to determine $\frac{dy}{dx}$. Since

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

we can isolate $\frac{dy}{dx}$ so that

$$\frac{dy}{dx} = \frac{12\cos t \sin^2 t}{-12 \sin t \cos^2 t} = -\frac{\cos t \sin^2 t}{\sin t \cos^2 t}.$$

Thus, when $t = \pi/4$, $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$ so that $\frac{dy}{dx} = -1$.

(c) It is tempting to write that

$$\frac{dy}{dx} = -\frac{\cos t \sin^2 t}{\sin t \cos^2 t} = -\frac{\sin t}{\cos t}.$$

However, this is not immediately true FOR ALL t, since we must check that we are not dividing by 0. In order to justify this step, we must compute the limits as t approaches the values where the denominator is 0. Thus, if $t = k\pi$ for $k \in \mathbb{Z}$ so that $\sin t = 0$, then

$$\lim_{t \to k\pi} \frac{\cos t \sin^2 t}{\sin t \cos^2 t} = \lim_{t \to k\pi} \frac{\sin t}{\cos t} = 0.$$

Further, if $t = k\pi + \pi/2$ for $k \in \mathbb{Z}$ so that $\cos t = 0$, then

$$\lim_{t \to k\pi + \pi/2} \frac{\cos t \sin^2 t}{\sin t \cos^2 t} = \lim_{t \to k\pi + \pi/2} \frac{\sin t}{\cos t} \quad \text{DNE}.$$

Thus, we see that $\frac{dy}{dx}$ does not exist when $\cos t = 0$. There are two values of t with $0 \le t \le 2\pi$ for which this is true: $t = \pi/2$ and $t = 3\pi/2$.

The value $t = \pi/2$ corresponds to the point $(4\cos^3(\pi/2), 4\sin^3(\pi/2)) = (0, 4)$.

The value $t = 3\pi/2$ corresponds to the point $(4\cos^3(3\pi/2), 4\sin^3(3\pi/2)) = (0, -4)$.

Note carefully the wording on page 231 of Stewart.

3. Let x be the distance traveled by *Titanic* in t hours; let y be the distance traveled by iceberg in t hours; and let z be the distance between *Titanic* and iceberg after t hours. We know that $\frac{dx}{dt} = 35$ km/h and $\frac{dy}{dt} = 25$ km/h. Thus, we must find $\frac{dz}{dt}$ when $t = 4$. These quantities are related by

$$(x + y)^2 + 100^2 = z^2.$$

Taking derivatives with respect to time gives

$$2(x + y)(\frac{dx}{dt} + \frac{dy}{dt}) + 0 = 2z \cdot \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{(x + y)(\frac{dx}{dt} + \frac{dy}{dt})}{z}.$$

Now, when $t = 4$, $x = 35 \cdot 4 = 140$, $y = 25 \cdot 4 = 100$, and $z = \sqrt{(140 + 100)^2 + 100^2} = 260$, so that

$$\frac{dz}{dt} = \frac{(140 + 100)(35 + 25)}{260} \approx 55.38$$

In conclusion, the distance between the *Titanic* and the iceberg is increasing at a rate of (approximately) 55.38 km/h at 4 p.m.

4. Differentiating both sides of $x^2y^2 - 6y + 2 = 0$ with respect to x gives

$$2xy^2 + 2x^2yy' - 6y' = 0,$$

where $y' = \frac{dy}{dx}$. We then solve for y' to find $y'(2x^2y - 6) = -2xy^2$, or

$$y' = \frac{2xy^2}{6 - 2x^2y}.$$

At the point $(x, y) = (2, 1)$, the slope is $y' = \frac{4}{6-8} = -2$. Thus, the equation for the tangent line at $(2, 1)$ is given by

$$y = -2(x-2) + 1 = -2x + 5.$$

5. If $f(x) = \sqrt[3]{x}$, then $L(x) = f'(a)(x-a) + f(a)$ so with $a = 27$ we get

$$L(x) = \frac{1}{3}(27)^{-2/3}(x-27) + \sqrt[3]{27} = \frac{1}{27}(x-27) + 3 = \frac{x}{27} + 2.$$

Thus, $f(28) \approx L(28)$ so that

$$\sqrt[3]{28} \approx \frac{28}{27} + 2 = 3\frac{1}{27}.$$

6.

(a) To determine the intervals on which f is increasing and the intervals on which f is decreasing, we need to see when $f'(x)$ is positive and when it is negative. Since $f'(x) = 0$ at $x = -1$ and $x = -3$, these are the only two critical points.

For $x < -3$, $f'(x) > 0$, so f is increasing.

For $-3 < x < -1$, $f'(x) < 0$, so f is decreasing.

For $x > -1$, $f'(x) > 0$, so f is increasing.

(b) To give the x-coordinate of any points at which f has a local maximum, and the x-coordinate of any points at which f has a local minimum we need to consider the critical values. At $x = -3$, $f'(x)$ switches from increasing to decreasing, so it is a local maximum. At $x = -1$, $f'(x)$ switches from decreasing to increasing, so it is a local minimum. (This is the First Derivative Test.)

(c) To determine the intervals of which f is concave up and the intervals on which f is concave down, we need to see when $f''(x)$ is positive and when it is negative. If we set $f''(x) = 0$ and solve for x, then

$$e^x \left(x + 3 - \sqrt{2}\right)\left(x + 3 + \sqrt{2}\right) = 0$$

when $\left(x + 3 - \sqrt{2}\right) = 0$ or $\left(x + 3 + \sqrt{2}\right) = 0$; equivalently, $f''(x) = 0$ when

$$x = -3 + \sqrt{2} \quad \text{or} \quad x = -3 - \sqrt{2}$$

since $e^x > 0$ for all values of x. We can then determine the intervals of concavity.

	e^x	$x+3-\sqrt{2}$	$x+3+\sqrt{2}$	$f''(x)$	$f(x)$
$x < -3-\sqrt{2}$	$+$	$-$	$-$	$+$	CU
$-3-\sqrt{2} < x < -3+\sqrt{2}$	$+$	$-$	$+$	$-$	CD
$x > -3+\sqrt{2}$	$+$	$+$	$+$	$+$	CU

Therefore f is concave up for $x < -3-\sqrt{2}$ and for $x > -3+\sqrt{2}$; f is concave down for $-3-\sqrt{2} < x < -3+\sqrt{2}$.

(d) By the above, we see that f changes concavity at $x = -3-\sqrt{2}$ and $x = -3+\sqrt{2}$ and therefore, these are the inflection points of f. *Note: It is not just enough to have $f''(x) = 0$ at an inflection point; you must also check that the concavity changes signs around that point.*

7.

By definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{3x^2\cos^2(1/x) - 0}{x - 0} = \lim_{x\to 0} 3x\cos^2(1/x).$$

In order to compute this limit we must use the Squeeze Theorem. For all θ, $-1 \le \cos\theta \le 1$. Thus, if $x \ne 0$,

$$0 \le \cos^2(1/x) \le 1.$$

If $x > 0$, then

$$0 \le 3x\cos^2(1/x) \le 3x,$$

so that

$$\lim_{x\to 0+} 3x\cos^2(1/x) = 0.$$

However, if $x < 0$, then

$$3x \le 3x\cos^2(1/x) \le 0,$$

so that

$$\lim_{x\to 0-} 3x\cos^2(1/x) = 0.$$

As both the one-sided limits are equal, we conclude that

$$f'(0) = \lim_{x\to 0} 3x\cos^2(1/x) = 0,$$

so that f is differentiable at 0.