Math 111 Prelim #1 Solutions - July 7, 2003

1.

(a) $\displaystyle \lim_{x\to -1} \frac{2x^2+x-1}{x+1} = \lim_{x\to -1} \frac{(x+1)(2x-1)}{x+1} = \lim_{x\to -1}(2x-1) = 2(-1)-1 = -3$

(b) $\displaystyle \lim_{x\to -1} \frac{2x^2+x-1}{2x+1} = \frac{2(-1)^2-1-1}{2(-1)+1} = \frac{0}{-1} = 0$

(c) $\displaystyle \lim_{x\to\infty}[\ln(x^2)-\ln x] = \lim_{x\to\infty}[2\ln x - \ln x] = \lim_{x\to\infty}\ln x = \infty$

(d) $\displaystyle \lim_{x\to -\infty} \frac{3x^3+2x-5x^{-2}}{2x^{-2}-x^2-2x^3} = \lim_{x\to -\infty} \frac{3+2x^{-2}-5x^{-5}}{2x^{-5}-x^{-1}-2} = \frac{3+0+0}{0-0-2} = -\frac{3}{2}$

2.

(a) f is not defined when we attempt to divide by 0, or take the square root of a negative number. Thus, the domain of f is

$$\mathscr{D}(f) = \{x > 1,\ x < -1\}.$$

(b) If we write $y = \frac{x}{\sqrt{x^2-1}}$ and solve for x we have $y\sqrt{x^2-1} = x$ or $y^2(x^2-1) = x^2$, so that $x = \sqrt{\frac{y^2}{y^2-1}}$. Hence,

$$f^{-1}(x) = \sqrt{\frac{x^2}{x^2-1}}.$$

(c) f^{-1} is not defined when we attempt to divide by 0, or take the square root of a negative number. Thus, the domain of f^{-1} is

$$\mathscr{D}(f^{-1}) = \{x > 1,\ x < -1\}.$$

(d) To find the horizontal asymptotes, we need to compute limits as $x \to \pm\infty$. Thus,

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-x^{-2}}} = \frac{1}{\sqrt{1-0}} = 1,$$

and

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{1-x^{-2}}} = \frac{-1}{\sqrt{1-0}} = -1.$$

To find the vertical asymptotes, we need to check the behaviour as x approaches -1 and 1 from both sides. Notice, however, that if $0 < x < 1$, f is not defined. Therefore the limit as $x \to 1-$ is not defined. Similarly, if $-1 < x < 0$, then the limit as $x \to -1+$ is not defined. Hence, we have

$$\lim_{x\to 1+} \frac{x}{\sqrt{x^2-1}} = \infty \quad \text{and} \quad \lim_{x\to -1-} \frac{x}{\sqrt{x^2-1}} = -\infty.$$

Thus the lines $y = 1$ and $y = -1$ are (one-sided) horizontal asymptotes, while the lines $x = 1$ and $x = -1$ are (one-sided) vertical asymptotes.

3. By definition, the slope of the tangent line is $f'(3)$. Hence,

$$\begin{aligned}
f'(3) &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} = \lim_{x\to 3}\frac{\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{3+1}}}{x-3} = \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2\sqrt{x+1}(x-3)} \\
&= \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2\sqrt{x+1}(x-3)}\,\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}} = \lim_{x\to 3}\frac{4-(x+1)}{2\sqrt{x+1}(x-3)} = \lim_{x\to 3}\frac{3-x}{2\sqrt{x+1}(x-3)} \\
&= \lim_{x\to 3}\frac{-1}{2\sqrt{x+1}} = -\frac{1}{4}
\end{aligned}$$

Therefore, the equation of the tangent line is $y - f(3) = f'(3)(x-3)$, or

$$y-\frac{1}{2} = -\frac{1}{4}(x-3).$$

4. In order for f to be continuous at $x=0$, we must have $\lim_{x\to 0} f(x) = f(0)$. Since $f(0)=0$, we need to compute the limit. As it stands, $\lim_{x\to 0} 3x^2\cos^2(1/x)$ is indeterminant. However, notice that $0 \le \cos^2(1/x) \le 1$ for $x \neq 0$. Thus,

$$0 \le 3x^2\cos^2(1/x) \le 3x^2.$$

Since $\lim_{x\to 0} 0 = 0$, and $\lim_{x\to 0} 3x^2 = 0$, the Squeeze Theorem tells us that

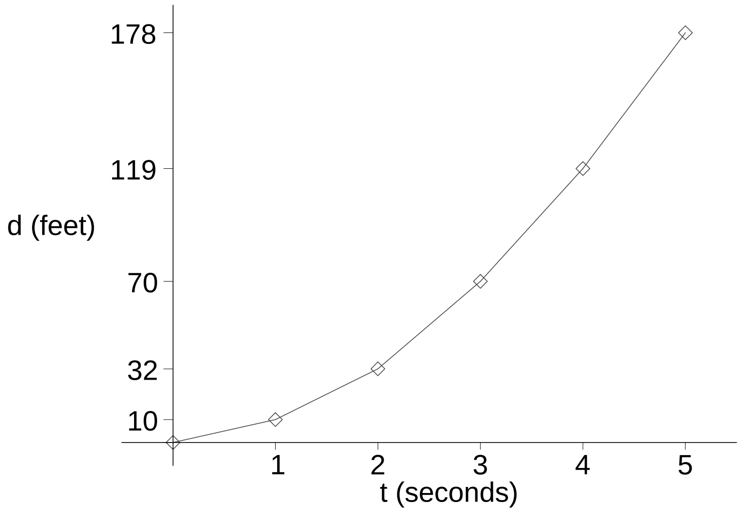
$$\lim_{x\to 0} 3x^2\cos^2(1/x) = 0.$$

As $\lim_{x\to 0} 3x^2\cos^2(1/x) = 0 = f(0)$ the definition of continuity *is* satisfied, so that f is continuous at 0.

5. This is similar to Section 2.4 #34. Let $f(x) = x^2 - 3$. Since f is a polynomial it is obviously continuous. Since $f(1) = -2$ and $f(2) = 1$, we can apply the Intermediate Value Theorem to conclude that there is a root c in the interval $(1,2)$. This root is called $\sqrt{3}$. (*Note*: If you continue tightening the interval, you can get a very good approximation for $\sqrt{3}$ as a sequence of decimal approximations. Indeed, the same argument gives $c \in (1.73204, 1.73206)$, etc.)

6.

(a) Yes, it is reasonable to assume that d is a continuous function of t, because it is unlikely for the distance traveled to suddenly jump from one value to another at an instant. Even as the car speeds up, slows down, backs up, or stops, the distance does not jump. The distance might increase or decrease rapidly, increase or decrease slowly, or remain constant. (It is not very likely that the car was "teleported" à la Star Trek.)



(b) Since velocity is the rate of change of position, we can estimate $v(3)$ by approximating $d'(3)$. The best we can do is to average the slopes of the two secant lines through $(3, 70)$, namely

$$\text{slope 1} = \frac{119 - 70}{4 - 3} = 49 \quad \text{and} \quad \text{slope 2} = \frac{70 - 32}{3 - 2} = 38.$$

Thus, $v(3) \approx 43.5$ feet/second.

(c) At $t = 3$, the velocity was approximately 43.5 feet/second. If we assume that this velocity remained constant until $t = 3.5$ seconds, then in the 0.5 seconds between $t = 3$ and $t = 3.5$, the car travelled $\frac{1}{2}(43.5) = 21.75$ feet. Thus,

$$d(3) \approx 70 + 21.75 = 91.75 \text{ feet.}$$

7.

(a) Since f is an odd function with $f(-2) = -1$, the definition of odd function tells us that $f(2) = 1$. Since $\lim_{x\to 2} f(x) = 1$, the definition of continuity is satisfied and f is necessarily continuous at $x = 2$.

(b) Although f is odd, and is continuous at $x = 2$, there is no reason that f must be continuous everywhere. For example, the function

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

is an odd function, has $f(-2) = -1$, and has $\lim_{x\to 2} f(x) = 1$, but has no roots in the interval $(-2, 2)$.