Math 111.01 July 17, 2003
Summer 2003

Maximum and Minimum Values (4.2)

Example. Determine the points at which $f(x) = \sin x$ attains its maximum and minimum.

Solution: $\sin x$ attains the value 1 whenever $x = \frac{\pi}{2} \pm 2\pi n$ and a minimum value of -1 whenever $x = \frac{3\pi}{2} \pm 2\pi n$, $n = 0, 1, 2, \ldots$.

Definition. The function f has an *absolute maximum* at c if $f(c) \geq f(x)$ for all $x \in \mathscr{D}(f)$.

Definition. The function f has an *absolute minimum* at c if $f(c) \leq f(x)$ for all $x \in \mathscr{D}(f)$.

Definition. The maximum and minimum values of f are called *extreme values*.

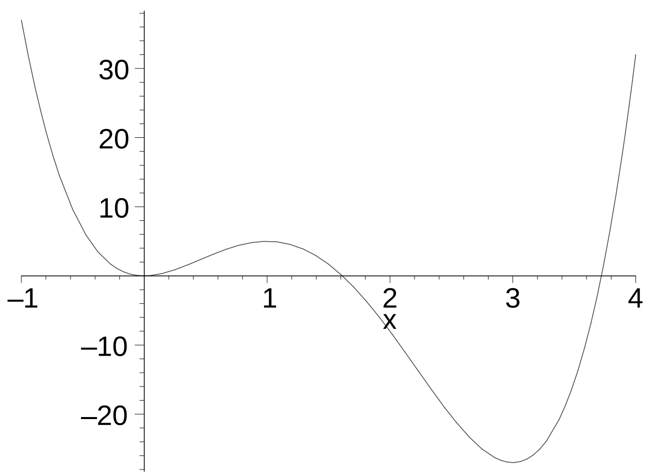
Example. Determine the extreme values of $f(x) = x^2$.

Solution: Since $x^2 \geq 0$ for all x, $f(x) \geq f(0)$. Therefore, $f(0) = 0$ is the absolute minimum.

However, f has no maximum.

Example. Graph $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$, and determine its absolute maximum and absolute minimum.

Solution: Graphically we see:



$\therefore$ absolute minimum: $f(3) = 27$, absolute maximum: $f(-1) = 37$

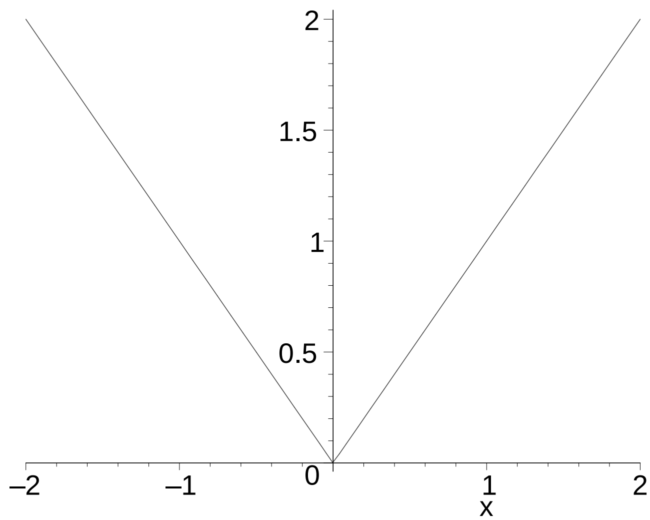
Fact (Extreme Value Theorem). If f is continuous on $[a, b]$, then f attains its absolute maximum $f(c)$ and its absolute minimum $f(d)$ at some numbers c, $d \in [a, b]$. That is, at points c and d with $a \leq c, d \leq b$.

Example. We need both continuity and a closed interval to guarantee extreme values.

Fact (Fermat's Theorem). If f has a local extrema at c and if $f'(c)$ exists, then $f'(c) = 0$.

Example. $f(x) = x^2$ has a local minimum at 0. Since $f'(x)$ exists, we must have $f'(0) = 0$. Indeed, $f'(x) = 2x$ so $f'(0) = 0$.

Example. Even though $f(x) = |x|$ has a local minimum at 0, we cannot use Fermat's Theorem since $f'(x)$ DNE at $x = 0$.



Definition. A *critical number* (or *critical value* or *critical point*) of a function f is a number c in $\mathscr{D}(f)$ such that either $f'(c) = 0$ or $f'(c)$ DNE.

Example. $f(x) = x^{-2}$ has no critical numbers. Even though $f'(x)$ DNE at $x = 0$, it is not a critcal number because $0 \notin \mathscr{D}(f)$.

Example. Find all critical numbers of $f(x) = x^{3/5}(4 - x)$.

Solution: By the product rule

$$f'(x) = \frac{3}{5}x^{-2/5}(4 - x) - x^{3/5} = \frac{12 - 8x}{5x^{2/5}}.$$

Thus $f'(x) = 0$ when $x = 12/8 = 3/2$, and $f'(x)$ DNE when $x = 0$. Since both $3/2$ and 0 are in $\mathscr{D}(f)$, they are both critical numbers.

$$\therefore \text{ CNs are } x = 3/2, \text{ and } x = 0.$$

Note. When finding CNs, it is imperative that you write $f'(x)$ in factored form.

Fact. If f has a local maximum or local minimum at c, then c is a critical number of f.

Closed Interval Method: To find the absolute maximum and the absolute minimum of a continuous function on a closed interval $[a, b]$:

(1) Find the values of f at the critcal numbers in (a, b).

(2) Find the values of f at the endpoints of $[a, b]$.

(3) You now have the maximum and minimum values.

Example. Find the absolute maximum and absolute minimum of $f(x) = x^4 - 2x^2 + 3$ on $[-2, 3]$.

Solution: (1) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$. Therfore the CNs are $x = 0$, $x = 1$, and $x = -1$. The function value at these three critical numbers are $f(0) = 3$, $f(1) = 2$, and $f(-1) = 2$.

(2) The function values at the endpoints are $f(-2) = 16 - 8 + 3 = 11$ and $f(3) = 81 - 18 + 3 = 66$.

By the Extreme Value Theorem, since f is continuous on the closed interval $[a, b]$, it must attain its absolute maximum and absolute minimum.

(3) By the closed interval method

$$\text{absolute maximum: } f(3) = 66$$
$$\text{absolute minimum: } f(1) = f(-1) = 2$$

Example. Find the absolute maximum and absolute minimum of $f(x) = x^2 e^{-x}$ on $[-1, 1]$.

Solution: $f'(x) = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2) = xe^{-x}(2 - x)$. Thus, the critical numbers are $x = 0$ and $x = 2$, and $f(0) = 0$, $f(2) = 4e^{-2}$.

HOWEVER, $x = 2$ is NOT in the given interval. Therefore we disregard it.

The values at the endpoints are $f(-1) = e$, $f(1) = e^{-1}$.

By the closed interval method

$$\text{absolute maximum: } f(-1) = e$$
$$\text{absolute minimum: } f(0) = 0$$

Derivatives and Shapes of Curves (4.3)

Recall from Section 2.10 that the derivative tells us information about the shape of a curve.

First Derivative

Fact. If $f'(c) = 0$, then f has a horizontal tangent at c.

Definition. f has a *critical number* at c in $\mathscr{D}(f)$ if either $f'(c) = 0$ or $f'(c)$ DNE.

Fact (Fermat's Theorem). If f has a local extremum at c, then c is a critical point of f.

Increasing/Decreasing Test

(a) If $f' > 0$ on an interval, then f is increasing on that interval.
(b) If $f' < 0$ on an interval, then f is dereasing on that interval.

First Derivative Test

Suppose that c is a critical number of the continuous function f.

(a) If f' changes sign from positive to negative at c, then f has a local maximum at c.
(b) If f' changes sign from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c, then f has no local extremum at c.

Hint: Remember all of these with a picture.

Second Derivative

Definition. A function f is *concave up* on an interval I if f' is increasing on I.

Definition. A function f is *concave down* on an interval I if f' is decreasing on I.

Definition. A point c where f changes concavity is called an *inflection point.*

Concavity Test

(a) If $f'' > 0$ on an interval, then f is concave up on that interval.
(b) If $f'' < 0$ on an interval, then f is concave down on that interval.

Second Derivative Test

Suppose that f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example. $f(x) = |x|$ has a local minimum at 0, but $f'(0)$ DNE.

Example. $f(x) = x^{1/3}$ changes concavity at 0 so that o is an inflection point, but $f'(0)$ and $f''(0)$ DNE.

Mean Value Theorem
Suppose that f is differentiable on (a, b) (and the one-sided derivatives exist at a and b).

Note that f MUST be continuous.

The secant line connecting a and b has slope

$$\frac{f(b) - f(a)}{b - a}.$$

Notice that there must be a point where the tangent is parallel to this secant.

Fact (Mean Value Theorem). If f is differentiable on $[a, b]$, then there exists a number c in (a, b) (that is, with $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$