Math 111.01 July 14, 2003
Summer 2003

Applications of the Chain Rule (3.5, 3.6, 3.7)

Tangents to Parametric Curves

Suppose that we have a parametric curve described by the equations $x = x(t)$ and $y = y(t)$. It is often possible to compute the equation of a tangent line at a point on the curve. By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

so that if $\frac{dx}{dt} \neq 0$, then we can write

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Example. Consider the circle $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t < 2\pi$. Find the equation of the tangent line when $t = \pi/4$.

Solution: Note that when $t = \pi/4$, that $x = 1/\sqrt{2}$, and $y = 1/\sqrt{2}$. Furthermore, $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$. Thus, when $t = \pi/4$,

$$\frac{dy}{dx} = \frac{-\sin(\pi/4)}{\cos(\pi/4)} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$$

so that the equation of the tangent line is

$$y - 1/\sqrt{2} = -1(x - 1/\sqrt{2}).$$

Implicit Differentiation

Consider the function f. We can represent this function as a formula $f(x)$. This function can also be represented by its graph $\{(x, y) : y = f(x)\}$. As a shortcut, we can write $y = f(x)$, and then compute $\frac{dy}{dx} = y' = f'(x)$.

Example. Suppose that

$$f(x) = e^x \sin(\sqrt{3x + x^{-2}}).$$

Let $y = f(x)$ and compute y'.

Solution: If $y = e^x \sin(\sqrt{3x + x^{-2}})$, then

$$\begin{aligned}
y' &= e^x \sin(\sqrt{3x + x^{-2}}) + e^x\,[\sin(\sqrt{3x + x^{-2}})]' \\
&= e^x \sin(\sqrt{3x + x^{-2}}) + e^x \cos(\sqrt{3x + x^{-2}})\,[\sqrt{3x + x^{-2}}]' \\
&= e^x \sin(\sqrt{3x + x^{-2}}) + e^x \cos(\sqrt{3x + x^{-2}})\,\frac{1}{2\sqrt{3x + x^-2}}\,[3x + x^{-2}]' \\
&= e^x \sin(\sqrt{3x + x^{-2}}) + e^x \cos(\sqrt{3x + x^{-2}})\,\frac{1}{2\sqrt{3x + x^-2}}\,(3 - 2x^{-3})
\end{aligned}$$

Example. Consider the graph given implicitly by $e^x - y = 7x$. If we write this as $y = e^x - 7x$, then we see this is the graph of the function $f(x) = e^x - 7x$. Now, we can take its derivative: $f'(x) = e^x - 7x$.

Alternatively, we can start directly with the equation $e^x - y = 7x$, and take the derivative with respect to x of both sides. Here, the "dee"-notation of Leibniz is useful.

$$\frac{d}{dx}(e^x - y) = \frac{d}{dx}(7x)$$

$$\frac{d}{dx}e^x - \frac{dy}{dx} = \frac{d}{dx}(7x)$$

$$e^x - \frac{dy}{dx} = 7$$

$$\frac{dy}{dx} = e^x - 7$$

Sometimes, the relationship between x and y does not define a function. In this case, it may still be possible to determine slopes of tangent lines to curves.

Example. Consider the circle $x^2 + y^2 = 1$. What is the equation of the tangent line to the circle at $(1/\sqrt{2}, 1/\sqrt{2})$?

Solution: This equation does not define a function. (Of course, we can consider the two functions $f_1(x) = \sqrt{1 - x^2}$ and $f_2(x) = -\sqrt{1 - x^2}$ for the top and bottom half, respectively, of the circle.) Taking derivatives of both sides with respect to x gives:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Thus, at the point $(1/\sqrt{2}, 1/\sqrt{2})$, the slope of the tangent is

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$$

and the equation of the tangent line is therefore

$$y - 1/\sqrt{2} = -1(x - 1/\sqrt{2}).$$

Question: At what points does the circle have a horizontal tangent? a vertical tangent?

This is an example of *implicit differentiation.*

Example. The equation $x^3+y^3=6xy$ describes a curve called the "Folium of Descartes." It is not possible to solve for y in terms of x. However, we can find the equation of various tangent lines. For example, the point $(3,3)$ lies on the Folium. Find the equation of the tangent line there.

Solution: Taking derivatives implicitly gives:

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$
$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}(6xy)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

Solving for $\frac{dy}{dx}$ gives

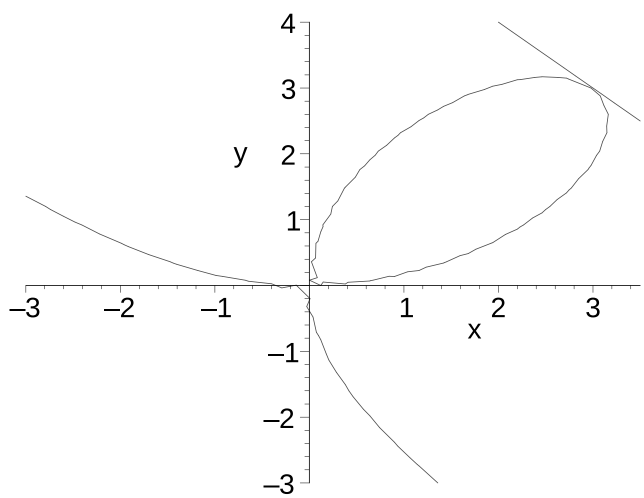
$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}.$$

Thus, at the point $(3,3)$, the slope of the tangent is

$$\frac{dy}{dx} = \frac{6(3)-3(3)^2}{3(3)^2-6(3)} = -1.$$

The equation of the tangent line is therefore

$$y-3 = -1(x-3).$$



When we combine implicit differentiation with the chain rule, we obtain a powerful technique for determining new derivative formulas.

Example. Compute $\frac{d}{dx}\ln x$.

Solution: If we write $y=\ln x$, then we can solve for y implicitly, and use a formula we know. That is, if $y=\ln x$, then $e^y=x$. Taking derivatives with respect to x of both sides gives:

$$\frac{d}{dx}e^y = \frac{dx}{dx}$$

$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$

And now the big ideas! $y = \ln x$, and $e^y = x$, so we substitute these back in and get

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

We can now prove the general power rule.

Example. Compute $\frac{d}{dx} x^a$ where $a \in \mathbb{R}$ (and $x \neq 0$ if $a < 0$).

Solution: If we write $y = x^a$, then we can solve for y implicitly, and use a formula we know. That is, if $y = x^a$, then $\ln y = a \ln x$. Taking derivatives with respect to x of both sides gives:

$$\frac{d}{dx} \ln y = \frac{d}{dx}(a \ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{a}{x}$$
$$\frac{dy}{dx} = \frac{ay}{x}$$

And now the big idea! $y = x^a$ so we substitute this back in and get

$$\frac{d}{dx} x^a = \frac{ax^a}{x} = ax^{a-1}.$$

Alternatively, we could write

$$y = x^a = e^{a \ln x}$$

so that

$$y' = e^{a \ln x} \frac{a}{x} = ax^{a-1}.$$

Question: Why is ln called the natural logarithm? What is so natural about the base $e = 2.71828\ldots$?

Example. Compute $\frac{d}{dx} \log_a x$ where $a > 0$.

Solution: If we write $y = \log_a x$, then $a^y = x$. Taking derivatives with respect to x of both sides gives:

$$\frac{d}{dx} a^y = \frac{dx}{dx}$$
$$a^y \ln a \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{a^y \ln a}$$

Now, we can substitute $y = \log_a x$ and $a^y = x$ to get

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Alternatively,

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{x\ln a}.$$

Thus, only in the case $a = e$, do the formulas work out nicely:

$$\frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}e^x = e^x \ln e = e^x.$$

Example. Compute $\frac{d}{dx}\sin^{-1} x$.

Solution: If we write $y = \sin^{-1} x$, then $\sin y = x$. Hence,

$$\frac{d}{dx}\sin y = \frac{dx}{dx}$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We now need to somehow discover what $\cos y$ is in terms of x if $\sin y = x$. Remember that $\sin^2 y + \cos^2 y = 1$. Therefore, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. Hence, $\cos y = \sqrt{1 - x^2}$, so that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$

Example. Compute $\frac{d}{dx}\tan^{-1} x$.

Solution: If we write $y = \tan^{-1} x$, then $\tan y = x$. Hence,

$$\frac{d}{dx}\tan y = \frac{dx}{dx}$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

We now need to somehow discover what $\sec y$ is in terms of x if $\sin y = x$. Remember that $\tan^2 y + 1 = \sec^2 y$. Therefore, $\sec^2 y = \tan^2 y + 1 = x^2 + 1$. Hence,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2 + 1}.$$

Homework. Compute $\frac{d}{dx}\cos^{-1} x$ and $\frac{d}{dx}\csc^{-1} x$.

Example. Find y' if $xy + \sin(x + y) = 3$.

Solution: Taking derivatives implicitly gives $y + xy' + \cos(x + y)(1 + y') = 0$ so that

$$y' = \frac{-\cos(x + y) - y}{x + \cos(x + y)}.$$

Now find y''. Note that it is easiest to work with the implicit equation involving y'.

Example. Compute $\frac{d}{dx}|x|$.

Solution: If we write $|x| = \sqrt{x^2}$, then

$$\frac{d}{dx}|x| = \frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}}2x = \frac{x}{|x|} = \frac{|x|}{x}.$$

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Example. Compute $\frac{d}{dx}\ln|x|$.

Solution: By the chain rule,

$$\frac{d}{dx}\ln|x| = \frac{1}{|x|}\frac{|x|}{x} = \frac{1}{x}.$$

Thus,

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

Logarithmic Differentiation

Sometimes, taking logs before taking derivatives allows us to simplify the calculations.

Example. Compute $\dfrac{d}{dx}\dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2\sin x)^5}$.

Solution: We could use the power, quotient, and chain rules together, but yuck! Instead, write

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2\sin x)^5}$$

and take logs:

$$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2\sin x)^5}\right) = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2\sin x).$$

Now, taking derivatives with respect to x gives

$$\frac{d}{dx}\ln y = \frac{3}{4}\frac{d}{dx}\ln x + \frac{1}{2}\frac{d}{dx}\ln(x^2+1) - 5\frac{d}{dx}\ln(3x+2\sin x)$$

so that

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{5(3+2\cos x)}{3x+\sin x},$$

or in other words

$$\frac{d}{dx}\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2\sin x)^5} = \left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2\sin x)^5}\right)\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15+10\cos x)}{3x+\sin x}\right).$$