Math 111.01 Summer 2003
July 16, 2003

The goal of this problem is to have you prove the chain rule.

Recall that the definition of derivative is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. If h is small, then we can approximate $f'(x)$ by $\dfrac{f(x+h) - f(x)}{h}$.

Suppose that we know $f(x)$ and $f'(x)$. We can use these to approximate the value of $f(x+h)$ for small h.

Solving $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$ for $f(x+h)$ yields $f(x+h) \approx f(x) + hf'(x)$. This is called a *linear approximation.*

Example: Suppose that we want to approximate the value of $(7.1)^2$. We know that $7^2 = 49$. We can use the idea above by letting $f(x) = x^2$. Then $f'(x) = 2x$. Use the formula above with $h = 0.1$ and $x = 7$. Thus, $f(x+h) \approx f(x) + hf'(x)$ or $(x+h)^2 \approx x^2 + 2hx$.

Therefore, $(7 + .1)^2 \approx 7^2 + 2(.1)(7) = 49 + 1.4 = 50.4$.

Use a linear approximation to prove the chain rule. That is, prove that $[f(g(x))]' = f'(g(x)) \cdot g'(x)$.

(Hint: Differentiate $f(g(x))$ using $g(x+h) \approx g(x) + hg'(x)$ and $f(z+k) \approx f(z) + kf'(z)$.)

SOLUTION:

By definition,

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Substituting $g(x+h) \approx g(x) + hg'(x)$ gives,

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h}.$$

Use $f(z+k) \approx f(z) + kf'(z)$ with $z = g(x)$ and $k = hg'(x)$, to yield

$$\begin{aligned} \lim_{h \to 0} \frac{f(g(x) + hg'(x)) - f(g(x))}{h} &= \lim_{h \to 0} \frac{f(g(x)) + hg'(x)f'(g(x)) - f(g(x))}{h} \\ &= f'(g(x))g'(x). \end{aligned}$$