Math 111.01 Summer 2003
Assignment #3 Solutions

**1.** Seriously, you should rework all of the problems on Prelim #1, paying special attention to those that you got incorrect.

**2.** Practice problems.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

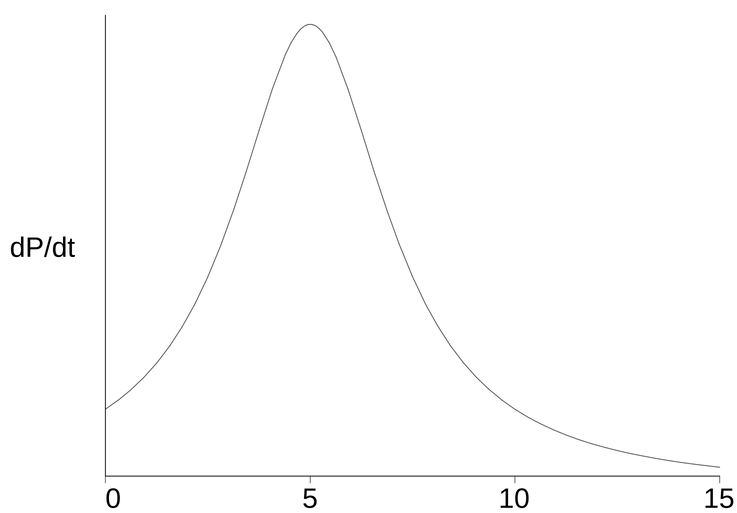
**3.** Practice computing derivatives.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

**4.** Problems to hand in.

**Section 2.8**

**#12.**
- $P'(0) \approx 0$ since $P(t)$ appears to have a horizontal tangent line at 0.
- $P'(t)$ seems to increase to 1 (approximately) at $t = 5$.
- $P'(t)$ decrease slowly from $t = 5$ to $4t = 10$ and continues to decrease for $t \geq 10$.
- $P'(t)$ is always positive. Since $P(t)$ "flattens" near $t = 15$, we have $P'(t)$ approaches 0 as $t$ increases.




As $t$ increases, we see that the rate of change of the yeast population approaches 0. That is, the yeast population stabilizes, and remains constant. (Just because $P'$ approaches 0, does NOT mean that $P$ approaches 0.)

**#32.** **a**. Recall that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. So $f(x)$ is discontinuous at $x = -2$: we have $\lim_{x\to -2} f(x)$ exists; however, $\lim_{x\to -2} f(x) \neq f(-2)$. $x = 0$ (the limit doesn't even exist) and $x = 5$ for the same reason. It's continuous at all other points in its domain.

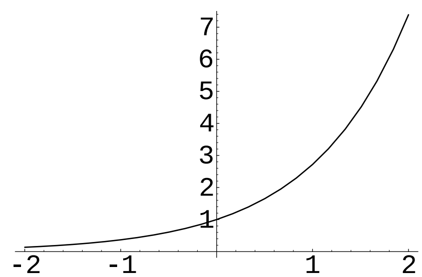
**b**. By Theorem 4, pg. 163, we know that if $f(x)$ is not continuous at $x = a$, then it is not differentiable there either. Immediately, we know $f$ is **not** differentiable at $x = -2, 1, 5$. Moreover, at $x = 2$, we can see there is no well-defined best line approximation (tangent line) to the graph: the graph is "pointy"( or "has a cusp") at $(2, g(2))$. So $g$ is not differentiable at $x = -2, 0, 2, 5$ and it is differentiable at all other points in its domain.

**#46.** Where is $f(x) = [\![x]\!]$ (the greatest integer function) not differentiable? A glance at the graph of $f(x)$ (see pg. 116) reveals that $f(x)$ is not even continuous at **any integer value** $a$. Recall that if a function is differentiable at a point $d$, then it must be continuous at $d$ (see Theorem 4, pg. 163).

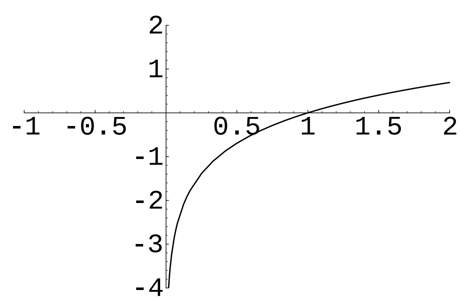
But for all non-integer values of $x$, it's clear that a best line approximation (a tangent line) exists at $(x, f(x))$ and that this tangent line is a horizontal line. Thus, $f'(x) = 0$, for all $x$ not an integer, while $f'(x)$ doesn't exist if $x$ is an integer.

**Section 2.10**

**#4.** **a**. Here is one example of a curve whose slope is always positive and increasing.

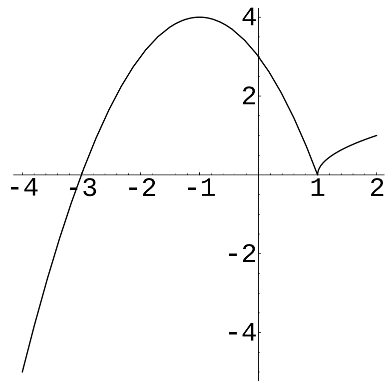



**b**. Here is one example of a curve whose slope is always positive and decreasing.




**c**. Here is one possibility for such a pair: $y = e^x$ has positive and increasing slope; $y = \ln(x)$ has positive and decreasing slope.

**#18.** Notice that $f$ is always concave down for $x \neq 1$ since $f'' < 0$. Also, in the interval $(-\infty, -1)$ the slope is positive, in $(-1, 1)$ the slope is negative, and in $(1, \infty)$ the slope is positive again. Since $f'(-1) = 0$, $f$ has a horizontal tangent at $-1$. We have a cusp at $x = 1$ since $f'(1)$ does not exist.



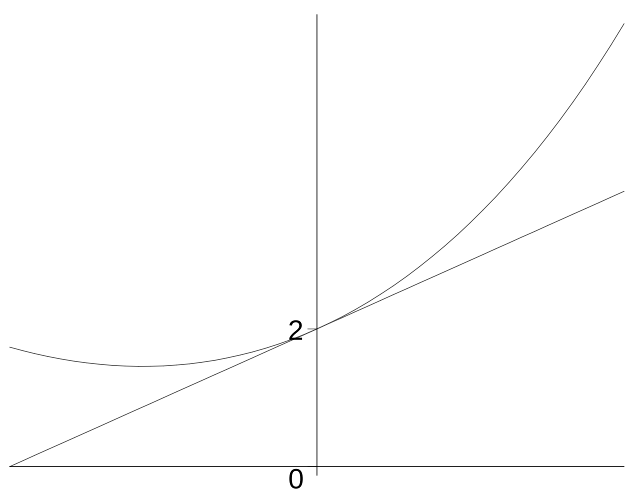

**#22.** $f'(x) = e^{-x^2}$ is always positive since $e^{-x^2} = \frac{1}{e^{x^2}} > 0$. Therefore, $f$ is always increasing.

**Section 3.1**

**#8.** $y = 5e^x + 3 \Rightarrow y' = 5e^x$

**#20.** $y = ae^v + bv^{-1} + cv^{-2} \Rightarrow y' = ae^v - bv^{-2} - 2cv^{-3} = ae^v - \frac{b}{v^2} - \frac{2c}{v^3}$

**#34.** $y' = 2x + 2e^x$. Therefore $y'(0) = 2$ and so the equation of the tangent line at $(0, 2)$ is $y = 2x + 2$.

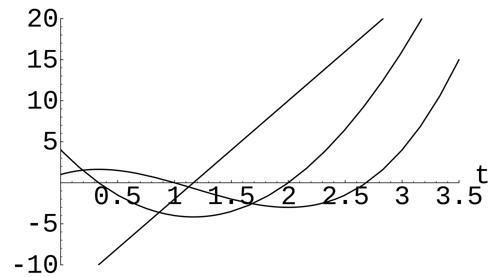



**#42.** $s = 2t^3 - 7t^2 + 4t + 1$

**a.** $v(t) = s'(t) = 6t^2 - 14t + 4$
$a(t) = v'(t) = s''(t) = 12t - 14$

**b**. $a(1) = 12 - 14 = -2\,\mathrm{m/s^2}$

**c**. Below is a plot of $s(t)$, $v(t)$, and $a(t)$.




**#46.** $f$ has a horizontal tangent when $f'(x) = 0$. Since $f'(x) = 6x^2 - 6x - 6$, setting it equal to zero gives $6(x^2 - x - 1) = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$ by the quadratic formula.

**Section 3.2**

**#4.** $g'(x) = \sqrt{x}e^x + e^x(\frac{1}{2}x^{-\frac{1}{2}}) = \sqrt{x}e^x + \frac{e^x}{2\sqrt{x}}$

**#10.** $H'(t) = e^t(6t + 20t^3) + e^t(1 + 3t^2 + 5t^4)$

**#28.** **a**. $(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$

**b**. $(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (4)(5) + (2)(-6) = 8$

**c**. $\left(\frac{f}{g}\right)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2} = \frac{(2)(-6) - (4)(5)}{4} = -8$

**d**. $\left(\frac{f}{f-g}\right)'(3) = \frac{[f(3) - g(3)]f'(3) - f(3)[f'(3) - g'(3)]}{[f(3) - g(3)]^2} = \frac{(4-2)(-6) - 4(-6-5)}{(4-2)^2} = 8$

**#36.** $f(x) = x^2e^x$ is concave down when $f'' < 0$.

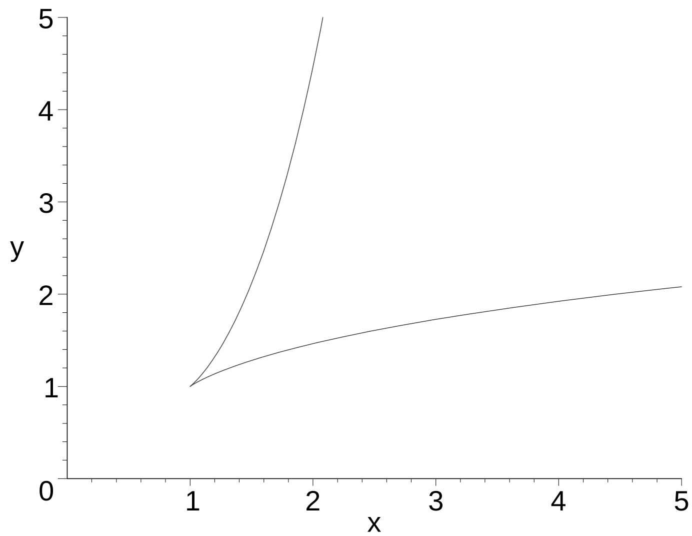
$$\begin{aligned}
f'(x) &= x^2e^x + 2xe^x \\
f''(x) &= (x^2e^x + 2xe^x) + (2xe^x + 2e^x) \quad \text{(product rule again)} \\
&\Rightarrow x^2e^x + 4xe^x + 2e^x = 0 \\
&\Rightarrow e^x(x^2 + 4x + 2) = 0 \\
&\Rightarrow x^2 + 4x + 2 = 0 \quad \text{since } e^x > 0 \text{ for all } x \\
&\Rightarrow x = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}
\end{aligned}$$

Therefore $f'' < 0$ when $x$ is in the interval $(-2 - \sqrt{2}, -2 + \sqrt{2})$.

**Section 1.7**

**#4.** $x = e^{-t} + t$ and $y = e^t - t$ for $-2 \le t \le 2$. We cannot eliminate the parameter easily to solve for $y$ in terms of $x$. However, if we plot points we can get an idea of what this curve looks like.

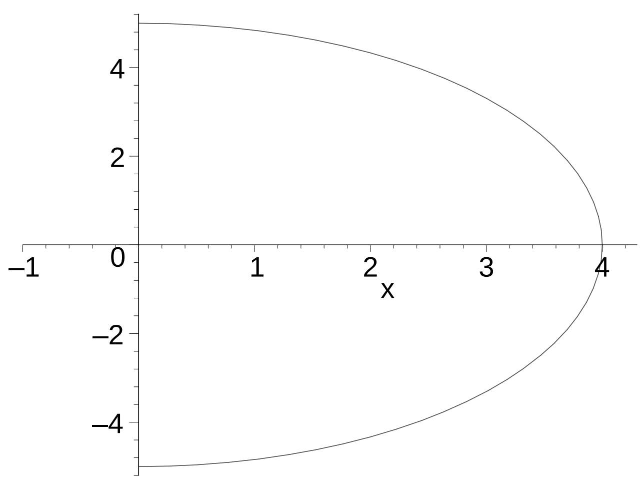
| $t =$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $x \approx$ | 5.4 | 1.7 | 1 | 1.4 | 2.1 |
| $y \approx$ | 2.1 | 1.4 | 1 | 1.7 | 5.4 |



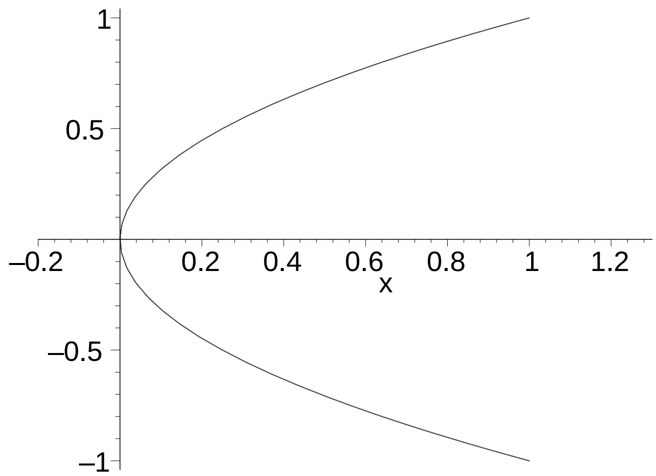

**#10.** **a**. If $x = 4\cos\theta$ and $y = 5\sin\theta$ for $-\pi/2 \le \theta \le \pi/2$, then $\cos\theta = x/4$ and $\sin\theta = y/5$. Squaring both $\sin\theta$ and $\cos\theta$, and adding them, gives

$$1 = (\cos\theta)^2 + (\sin\theta)^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2.$$

**b**. This is the equation of a (half)-ellipse. The parametric curve starts at $(0,-5)$ and traverses the ellipse through $(4,0)$ to $(0,5)$.



**#18.** If $x = \cos^2 t$ and $y = \cos t$ for $0 \le t \le 4\pi$, then $x = y^2$ for $0 \le x \le 1$ and $-1 \le y \le 1$. Thus, this describes a parabola starting at $(1, 1)$, going through the origin to $(1, -1)$, going back through the origin to $(1, 1)$, going through the origin again to $(1, -1)$, and finally going through the origin and ending at $(1, 1)$.




**Section 3.4**

**#8.** If $y = \frac{\sin x}{1+\cos x}$, then

$$y' = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

**#14.**

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{0\cdot\cos x - 1\cdot(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

**#18.** If $f(x) = e^x \cos x$, then $f'(x) = e^x \cos x - e^x \sin x$. Hence, $f'(0) = 1$. Thus, the equation of the tangent line is

$$y - 1 = 1(x - 0) \quad \text{or} \quad y = x + 1.$$

**#26.** In order to find the points on the curve $y = \frac{\cos x}{2+\sin x}$ at which the tangent is horizontal, we need to find the points $x$ at which $y' = 0$.

Hence,

$$y' = \frac{(2+\sin x)(-\sin x) - \cos x \cos x}{(2+\sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

Notice that $(2+\sin x)^2 \neq 0$ so that $y'$ is defined for all $x$. Thus, $y' = 0$ when $-2\sin x - 1 = 0$ or $\sin x = -1/2$. However, there are two "reference values" of $x \in [0, 2\pi)$ with $\sin x =$

$-1/2$; namely $x = 7\pi/6$ and $x = 11\pi/6$. Of course, sin is $2\pi$-periodic, so there are infinitely many values of $x$ at which $\sin x = -1/2$. Thus, $y$ has a horizontal tangent when

$$x = \frac{7\pi}{6} + 2\pi n, \quad \text{and} \quad x = \frac{11\pi}{6} + 2\pi n, \quad n \in \mathbb{Z}.$$

If $x = \frac{7\pi}{6} + 2\pi n$, then $y = -1/\sqrt{3}$, and if $x = \frac{11\pi}{6} + 2\pi n$, then $y = 1/\sqrt{3}$. Thus, $y$ has a horizontal tangent at the points

$$(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}), \quad \text{and} \quad (\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}), \quad n \in \mathbb{Z}.$$

## 5. More practice computing derivatives.

**Section 2.8**

**#20.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{5 - 4(x+h) + 3(x+h)^2 - (5 - 4x + 3x^2)}{h} \\
&= \lim_{h\to 0} \frac{5 - 4x - 4h + 3x^2 + 6xh + 3h^2 - 5 + 4x - 3x^2}{h} \\
&= \lim_{h\to 0} \frac{-4h + 6xh + 3h^2}{h} \\
&= \lim_{h\to 0} (-4 + 6x + 3h) \\
&= -4 + 6x
\end{aligned}$$

Thus, $\mathscr{D}(f) = \mathbb{R}$ and $\mathscr{D}(f') = \mathbb{R}$.

**#22.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h) + \sqrt{x+h} - (x + \sqrt{x})}{h} = \lim_{h\to 0} \frac{h}{h} + \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= 1 + \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = 1 + \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= 1 + \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = 1 + \frac{1}{2\sqrt{x}} = 1 + \frac{1}{2}x^{-1/2}
\end{aligned}$$

Thus, $\mathscr{D}(f) = \{x \geq 0\}$ and $\mathscr{D}(f') = \{x > 0\}$.

**#23.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{1 + 2(x+h)} - \sqrt{1+2x}}{h} = \lim_{h\to 0} \frac{\sqrt{1 + 2(x+h)} - \sqrt{1+2x}}{h} \cdot \frac{\sqrt{1 + 2(x+h)} + \sqrt{1+2x}}{\sqrt{1 + 2(x+h)} + \sqrt{1+2x}} \\
&= \lim_{h\to 0} \frac{1 + 2x + 2h - 1 - 2x}{h(\sqrt{1 + 2(x+h)} + \sqrt{1+2x})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{1 + 2(x+h)} + \sqrt{1+2x})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{1 + 2(x+h)} + \sqrt{1+2x}} = \frac{2}{\sqrt{1+2x} + \sqrt{1+2x}} \\
&= \frac{1}{\sqrt{1+2x}}
\end{aligned}$$

Thus, $\mathscr{D}(f) = \{x \geq -1/2\}$ and $\mathscr{D}(f) = \{x > -1/2\}$.

**#24.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{3+(x+h)}{1-3(x+h)} - \frac{3+x}{1-3x}}{h} \\
&= \lim_{h\to 0} \frac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)} \\
&= \lim_{h\to 0} \frac{3+x+h-9x-3x^2-3xh-3+9x+9h-x+3x^2+3xh}{h(1-3x-3h)(1-3x)} \\
&= \lim_{h\to 0} \frac{h+9h}{h(1-3x-3h)(1-3x)} \\
&= \lim_{h\to 0} \frac{10}{(1-3x-3h)(1-3x)} \\
&= \frac{10}{(1-3x)^2}
\end{aligned}$$

Thus, $\mathscr{D}(f) = \{x \neq 1/3\}$ and $\mathscr{D}(f') = \{x \neq 1/3\}$.

**Page 259**

**#3.**

$$y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3}$$

**#6.**

$$y' = \frac{e^x(1+x^2)-2xe^x}{(1+x^2)^2} = \frac{e^x(x^2-2x+1)}{(1+x^2)^2} = \frac{e^x(x-1)^2}{(1+x^2)^2}$$

**#9.**

$$y' = \frac{(1-t^2)-t(-2t)}{(1-t^2)^2} = \frac{t^2+1}{(1-t^2)^2}$$

**Section 3.4**

**#4.**

$$y' = \sec t \tan t + \sec^2 t$$

**#9.**

$$y' = \frac{(\sin x + \cos x) - x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{\sin x + \cos x - x\cos x + x\sin x}{(\sin x + \cos x)^2}$$

**#11.**

$$y' = \sec\theta\tan\theta\tan\theta + \sec\theta\sec^2\theta = \sec\theta(\tan^2\theta + \sec^2\theta)$$

## 6.

**a**. To compute $f'(0)$, use the definition.

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2|\cos\frac{\pi}{2x}| - 0}{x-0} = \lim_{x\to 0} x|\cos\frac{\pi}{2x}|$$

Now, in order to compute this limit, we need the Squeeze Theorem. Since $-1 \leq \cos\theta \leq 1$ for all $\theta$, we have $0 \leq |\cos\theta| \leq 1$. Thus, if $x > 0$, then

$$x \cdot 0 \leq x|\cos\frac{\pi}{2x}| \leq 1 \cdot x$$

However, if $x < 0$, then (because we have a negative number, the inequalities switch)

$$x \cdot 0 \geq x|\cos\frac{\pi}{2x}| \geq 1 \cdot x$$

Since $\lim_{x\to 0} 0 = 0$, and $\lim_{x\to 0} x = 0$, the first inequalities give us $\lim_{x\to 0+} x|\cos\frac{\pi}{2x}| = 0$, while the second inequalities give us $\lim_{x\to 0-} x|\cos\frac{\pi}{2x}| = 0$. Together, they tell us

$$\lim_{x\to 0} x|\cos\frac{\pi}{2x}| = 0$$

so that $f'(0) = 0$.

**b.** To show $f'(1/3)$ does not exist, we attempt to compute $\lim_{x\to 1/3}\frac{f(x)-f(1/3)}{x-1/3}$. Note that $f(1/3) = 0$. Thus,

$$\lim_{x\to 1/3}\frac{f(x)-f(1/3)}{x-1/3} = \lim_{x\to 1/3}\frac{x^2|\cos\frac{\pi}{2x}|}{x-1/3}.$$

Attempting to plug in 1/3 gives the indeterminant form $\left[\frac{0}{0}\right]$. Since we cannot factor, we are left to use a calculator.

(i) Plot a graph of $f(x) = x^2|\cos\frac{\pi}{2x}|$ and zoom in on $x = 1/3$. The graph looks like a cusp. This leads us to suspect the derivative DNE.

(ii) Plot a graph of $\frac{x^2|\cos\frac{\pi}{2x}|}{x-1/3}$ and zoom in on $x = 1/3$. The graph looks like a vertical line. This tells us the "tangent at 1/3 is vertical." That is, $f'(1/3)$ DNE.

(iii) Confirm this with a table of values for the above.

**7.** Currently, we do not have techniques that allow us to determine $\lim_{x\to 0}(\sec x)^{1/x^2}$. However, we can approximate it with a calculator.

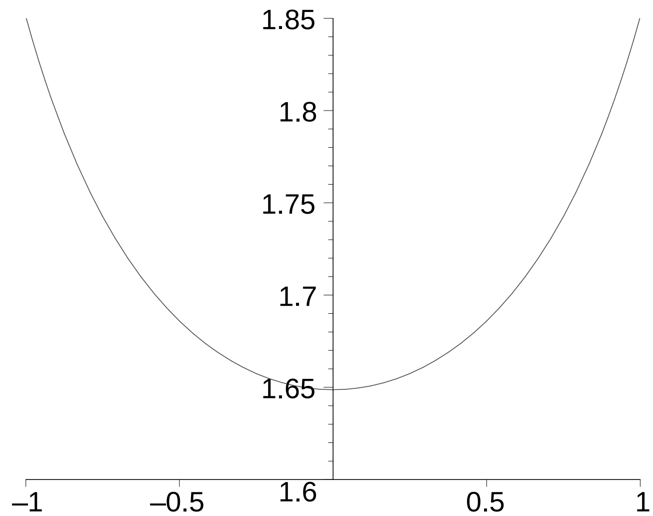
If we use a table of values on the TI-83, then we get the following

| $x$ | $(\sec x)^{1/x^2}$ |
|---|---|
| 0.0001 | 1.6487 |
| -0.0001 | 1.6487 |
| 0.00001 | 1.6487 |
| -0.00001 | 1.6487 |
| 0.000001 | 1.6482 |
| -0.000001 | 1.6482 |
| 0.0000000001 | 1.0000 |
| -0.0000000001 | 1.0000 |
| 0.00000000000001 | 1.0000 |
| -0.00000000000001 | 1.0000 |

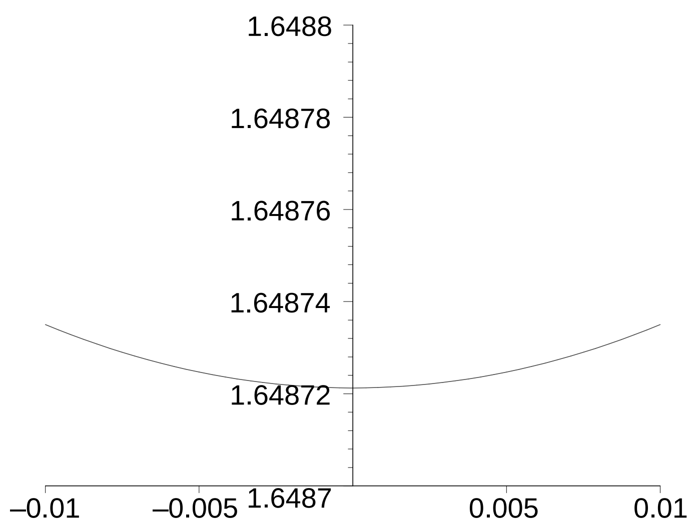
This leads us to suspect that

$$\lim_{x \to 0} (\sec x)^{1/x^2} = 1.$$

If we graph $(\sec x)^{1/x^2}$ then the graph appears to be parabolic, and going through 1.65.



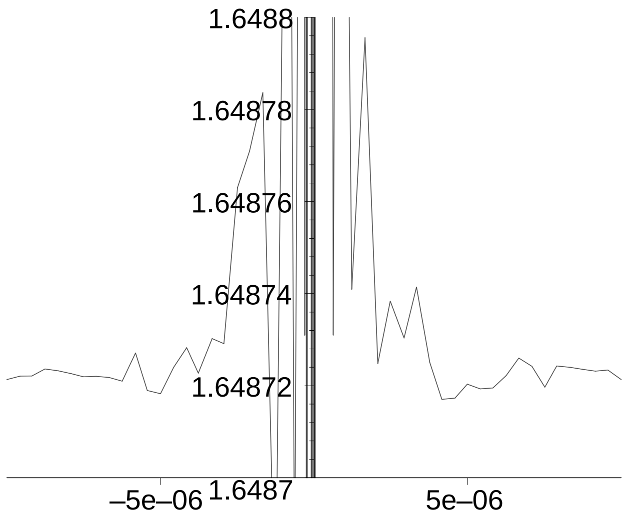

However, if we zoom in near $x = 0$, it appears to be nearly linearly, and going through 1.64872.




Thus, graphing leads us to guess that

$$\lim_{x \to 0} (\sec x)^{1/x^2} \approx 1.64872.$$

But, if we zoom in even more (using the computer software Maple), we see crazy behaviour!




This graph leads us to guess that

$$\lim_{x \to 0} (\sec x)^{1/x^2} \quad \text{DNE.}$$

These results are in conflict! Later, we will see how to do this algebraically.