Math 111.01 Summer 2003
Assignment #2 Solutions

1. Practice problems.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.
Section 2.4 #34 Answer: Let $f(x) = x^2 - 2$. Since f is a polynomial it is obviously continuous. Since $f(0) = -2$ and $f(2) = 2$, we can apply the Intermediate Value Theorem to conclude that there is a root c in the interval $(0, 2)$. This root is called $\sqrt{2}$. (*Note*: If you continue tightening the interval, you can get a very good approximation for $\sqrt{2}$ as a sequence of decimal approximations. Indeed, the same argument gives $c \in (1.41420, 1.41422)$, etc.)

2. Problems to hand in.

Section 2.2

#4. **a**. $\lim_{x\to 0} f(x) = 1$

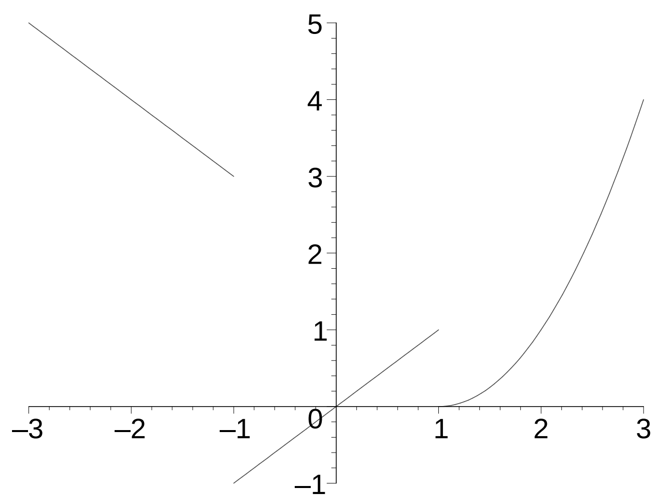
b. $\lim_{x\to 3-} f(x) = 4$

c. $\lim_{x\to 3+} f(x) = 2$

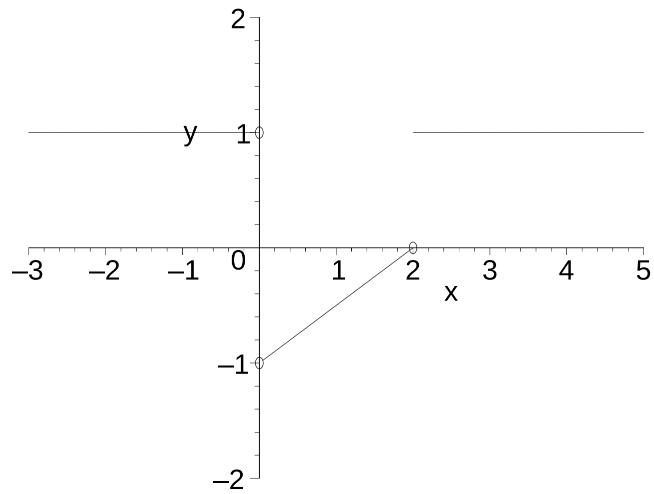
d. $\lim_{x\to 3} f(x)$ does not exist because the one-sided limits in (b) and (c) are not equal.

e. $f(3) = 3$

#6. $\lim_{x\to a} f(x)$ exists for all a except for $a = 1$ and $a = -1$.



#10. $\lim_{x\to 0-} f(x) = 1$, $\lim_{x\to 0+} f(x) = -1$, $\lim_{x\to 2-} f(x) = 0$, $\lim_{x\to 2+} f(x) = 1$, $f(2) = 1$, $f(0)$ is undefined.



Section 2.3

#8. **a**. The left side of the equation is not defined for $x = 2$, but the right side is.

b. Since the equation holds for all $x \neq 2$, it follows that both sides of the equation apporoach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim_{x \to a} f(x)$, we never consider $x = a$.

#14.

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{1+h}-1)\cdot(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

#20.

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \lim_{t\to 0} \frac{t^2+t-t}{t(t^2+t)} = \lim_{t\to 0} \frac{t^2}{t(t^2+t)} = \lim_{t\to 0} \frac{1}{t+1} = \frac{1}{0+1} = 1$$

#30. If $x > 2$, then $|x-2| = x-2$ so that

$$\lim_{x\to 2+} \frac{|x-2|}{x-2} = \lim_{x\to 2+} \frac{x-2}{x-2} = \lim_{x\to 2+} 1 = 1.$$

If $x < 2$, then $|x-2| = -(x-2)$ so that

$$\lim_{x\to 2-} \frac{|x-2|}{x-2} = \lim_{x\to 2-} \frac{-(x-2)}{x-2} = \lim_{x\to 2-} -1 = -1.$$

The right and left limits are different; hence

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

does not exist.

#38.

$$\lim_{v \to c-} \left(L_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} \right) = L_0\sqrt{1-1} = 0$$

As the velocity approaches the speed of light, the length approaches 0. Note that a limit from the left is necessary since L is not defined for $v > c$.

Section 2.4

#8. Answers may vary a bit on some of these depending on one's assumptions, but here are some reasonable answers to the questions.

- **a**. It is continuous because the temperature at the location changes smoothly with time; there are not instantaneous jumps between temperatures.
- **b**. It is continuous because the temperature at one particular time varies smoothly without jumping as you change how far you are from New York City.
- **c**. It could be discontinuous because as you move west from New York City, you might encounter a jump in altitude because of, say, a cliff.
- **d**. This function is discontinuous because as you travel more distance, the cost of the ride jumps in small intervals (at best, the cost would increase a single cent but never fractions of cents).
- **e**. This is a little tricky, but if you assume that current changes between 0 and some large value instantaneously when you flip on the switch, the function would be discontinous. If you adopt a different perspective, you may be able to defend the idea that the function is continuous.

#19. We want to show that $f(x) = e^x \sin 5x$ is continuous on its domain (which is all real numbers). By Theorem 7, e^x is continuous on all reals. Now $5x$ and $\sin x$ are continuous by Theorem 7, and the composite function, $(\sin x) \circ (5x) = \sin 5x$ is continuous everywhere by Theorem 9. The product of these two functions, $e^x \sin 5x$, is continuous on $(-\infty, \infty)$ by Theorem 4, #4.

#30. Since we're dealing with rational functions, each portion of F is continuous on its domain by Theorem 5. We need only check what happens at $r = R$. Let's compute the limit of F as r approaches R in either direction:

$$\lim_{r \to R^-} F(r) = \frac{GMR}{R^3} = \frac{GM}{R^2}, \text{ and } \lim_{r \to R^+} F(r) = \frac{GM}{R^2} \text{ also,}$$

so $\lim_{r \to R} F(r) = \frac{GM}{R^2}$. We know that $F(R) = \frac{GM}{R^2}$ as well, so F must be continuous at R. Hence F is a continuous function of r.

#38. Let $f(x) = \ln x - e^{-x}$; then f is continuous on $[1, 2]$ by Theorems 7 and 4. We want to show that $f(c) = 0$ for some c in the interval $(1, 2)$. Note that $f(1) = 0 - e^{-1} < 0$, and $f(2) = \ln 2 - e^{-2} > 0$, so since $f(1) < 0 < f(2)$, by the Intermediate Value Theorem, there is some c in $(1, 2)$ such that $f(c) = 0$. Thus $\ln c - e^{-c} = 0$, so $\ln c = e^{-c}$.

#45. If a number x is exactly one more than its cube, then $x = x^3 + 1$, or equivalently $x - 1 = x^3$. Let $f(x) = x^3 - x + 1$; then x is one more than its cube if and only if $f(x) = 0$. Since f is a polynomial, it is continuous everywhere. We can compute that $f(-2) = -8 + 2 + 1 = -5$ and $f(0) = 0 - 0 + 1 = 1$, so $f(-2) < 0 < f(0)$. Thus by the Intermediate Value Theorem, f has a root c in $(-2, 0)$, and so there is some number c in that interval that is one more than its cube.

Section 2.5

#14. Find the limit: $\lim_{x\to 5^-} \frac{e^x}{x-5}$. Notice that as x goes to 5 that $x - 5$ goes to 0. Indeed, as x goes to 5 from the **left** side, $x - 5$ values are small in absolute value and **negative**. Meanwhile, as x goes to 5 we have e^x gets close to e^5, a positive number. Quotients of numbers near e^5 over very small negative values will tend to $-\infty$. Thus,

$$\lim_{x\to 5^-} \frac{e^x}{x-5} = -\infty.$$

(Graphing the function on a graphics calculator reveals that indeed the graph of the function has a vertical asymptote as $x = 5$ and that the graph "shoots down" from the left side.)

#24. Find the limit:

$$\lim_{x\to\infty} \frac{\sin^2(x)}{x^2}.$$

From basic trig that for all u, we have $-1 \le \sin(u) \le 1$. So, for positive u, we have

$$|\frac{\sin(u)}{u}| \le \frac{1}{u}, \qquad (*)$$

which goes to 0 as u gets large. Recall that $\sin^2(x)$ means $(\sin(x))^2$, rather than $\sin(x^2)$. Notice that $0 \le \sin^2(x) \le 1$; hence, $\frac{\sin^2(x)}{x^2}$ goes to 0 as x gets large. So, here is the solution:

$$\lim_{x\to\infty} \frac{\sin^2(x)}{x^2} = 0.$$

(We could have used law #6 (with $n = 2$), on page 112, to get from $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ to $\lim_{x\to\infty} \frac{\sin^2(x)}{x^2} = 0$.)

(Notice that $(*)$ shows that $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$, a limit that is useful to remember. What is $\lim_{x\to-\infty} \frac{\sin(x)}{x}$?)

#28. Find the limit: $\lim_{x\to\infty} e^{-x^2}$.

As x takes on large positive values, notice that e^x goes to ∞. (See Figure 16, page 138.) Thus, as x gets large, $e^{-x} = \frac{1}{e^x}$ goes to 0. That is, $\lim_{x\to\infty} e^{-x} = 0$.

We have $\frac{1}{e^{x^2}} = e^{-x^2}$. For $x \ge 1$, we have $0 \le e^{-x^2} \le e^{-x} = \frac{1}{e^x}$. The first and last expressions go to 0 as x goes to infinity; hence, we have $\lim_{x\to\infty} e^{-x^2} = 0$ also (by the "infinite" version of the Squeeze Theorem!)

Section 2.6

#6. Let $f(x) = x^2$. Find the slope of the tangent to $f(x)$ at $(-1, 1)$, in two ways.

a. (i) By using Definition 1 : $m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.
Substituting x^2 for $f(x)$ and -1 for a, we have

$$m = \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1}(x-1) = -2$$

So the slope of the tangent line in question is -2.
(ii) By using Equation 2: $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.
Substituting x^2 for $f(x)$ and -1 for a leads to

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(-1+h)^2-1}{h} = \lim_{h\to 0} \frac{1-2h+h^2-1}{h} = \lim_{h\to 0} \frac{-2h+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(-2+h)}{h} = \lim_{h\to 0}(-2+h) = -2. \end{aligned}$$

(Thankfully we're in agreement with the result we got using Definition 1!)

b. Since we want the tangent at $(-1, -1)$, and we found the slope (in two ways) to be -2 at this point, the equation of the tangent line is

$$y - (-1) = -2(x - (-1)) \quad \text{or equivalently,} \quad y = -2x - 1.$$

#12. Let $f(x) = \frac{1}{\sqrt{x}}$.

a. Find the slope to $f(x)$ at $x = a$.

Let's use Definition 2. We have

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}}}{h}$$

(We'll simplify a complex fraction by multiplying top and bottom by the product of the denominators up in the present numerator)

$$= \lim_{h\to 0} \frac{\sqrt{a}-\sqrt{a+h}}{h(\sqrt{a+h})(\sqrt{a})}$$

(We'll multiply by the conjugate of the expression in the current numerator, multiplying in both numerator and denominator.)

$$\begin{aligned} &= \lim_{h\to 0} \frac{(\sqrt{a}-\sqrt{a+h})(\sqrt{a}+\sqrt{a+h})}{h(\sqrt{a+h})(\sqrt{a})(\sqrt{a}+\sqrt{a+h})} = \lim_{h\to 0} \frac{-h}{h(\sqrt{a+h}\sqrt{a})(\sqrt{a}+\sqrt{a+h})} \\ &= \lim_{h\to 0} \frac{-1}{(\sqrt{a+h}\sqrt{a})(\sqrt{a}+\sqrt{a+h})} = -\frac{1}{a(2\sqrt{a})} = -\frac{1}{2a^{\frac{3}{2}}} = -\frac{1}{2}a^{-\frac{3}{2}} \end{aligned}$$

b. Find slopes of tangent lines at $(1,1)$ and $(4,\frac{1}{2})$ respectively.

For $(1,1)$, just evaluate the expression in the first part at $a = 1$ to get $m = -\frac{1}{2}$. Similarly, for $(4,\frac{1}{2})$, we get $m = -\frac{1}{16}$.

c. Simply graph $f(x) = \frac{1}{\sqrt{x}}$ and tangent lines on your graphics calculator. You've determined the slopes of the tangent lines; use the point-slope form to determine the equation of the two tangent lines. The equations of the tangent lines : at $(1,1)$ the tangent line is $y = -\frac{1}{2}x + \frac{3}{2}$, and at $(4,\frac{1}{2})$ it is $y = -\frac{1}{16}x + \frac{9}{2}$.

#16. Let $H(t) = 58t - 0.83t^2$, the height of the arrow on the moon.

a. Find velocity at $t = 1$. To find the velocity at $t = 1$, we need to compute

$$v(1) = H'(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h}.$$

Hence, we have

$$\begin{aligned}\lim_{h\to 0} \frac{H(1+h) - H(1)}{h} &= \lim_{h\to 0} \frac{(58(1+h) - 0.83(1+h)^2) - (58 - 0.83))}{h} \\ &= \lim_{h\to 0} \frac{h(58 - 0.83(2+h)}{h} \\ &= \lim_{h\to 0} 958 - 0.83(2+h)) \\ &= 58 - 0.83(2) = 56.34\end{aligned}$$

So, the velocity at time $t = 1$ is given by

$$v(1) = 56.34 \text{ m/s}.$$

b. Find the velocity at $t = a$. We need to determine $v(a) = H'(a) = \lim_{h\to 0} \frac{H(a+h)-H(a)}{h}$. We have that

$$v(a) = \lim_{h\to 0} \frac{(58(a+h) - 0.83(a+h)^2) - (58a - 0.83a^2))}{h} = \lim_{h\to 0} (58 - 0.83(2a+h)).$$

Now

$$v(a) = 58 - 1.66a \text{ (meters/second)},$$

the velocity at $t = a$.

c. When will the arrow hit the moon? That is, for time t do we have $H(t) = 0$? Solving, we get $0 = H(t) = 58t - .83t^2 = t(58 - .83t)$. We are not interested in the $t = 0$ solution, since we sure knew that the arrow was at height 0 when $t = 0$. The other solution interests us: $t = 69.88$ seconds approximately.

d. With what velocity will the arrow hit the moon? We know that it hits the moon at approximately $t = 69.88$ seconds. Its velocity at time a is given above $(58 - 1.66a)$. Substituting we get the velocity is $58 - 1.66(69.88) = -58.001$ meters/second. (Can you explain why the velocity is negative?)

Section 2.7

#18. Let $f(x) = \sqrt{3x+1}$. Find $f'(a)$.

We have $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. We simplify the expression inside the limit first. We have $\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{3(a+h)+1}-\sqrt{3a+1}}{h}$.
After multiplying by the conjugate of the numerator, we get

$$\frac{3h}{h(\sqrt{3(a+h)+1}+\sqrt{3a+1})}.$$

Thus,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{3}{2(\sqrt{3a+1})} = \frac{3}{2}(3a+1)^{-\frac{1}{2}}.$$

#22. Consider $\lim_{x\to\frac{\pi}{4}} \frac{\tan(x)-1}{x-\frac{\pi}{4}}$. Find the function $f(x)$ and value $x = a$ such that the expression above is its derivative at a.

Answer: $f(x) = \tan(x)$ and $a = \pi/4$ since

$$\lim_{x\to\frac{\pi}{4}} \frac{\tan(x)-1}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\tan(x)-\tan(\frac{\pi}{4})}{x-\frac{\pi}{4}}.$$

#34. In this problem we're only given enough information to **approximate** the $E'(t)$. (We can't take a limit since we don't have information about the function, such as the values of the functions of points arbitrarily close to the times in question.)

To approximate, we average two secant lines slopes, taken in symmetric intervals around the point in question.

We now estimate $E'(1910)$ by calculating the average slopes of two secant lines in symmetric intervals around $t = 1910$. $\frac{E(1910)-E(1900)}{10}$ and $\frac{E(1920)-E(1910)}{10}$ are the two slopes of secants we'll use. Computing these from the data in the table and then averaging them gives us: (after appropriate simplifications) .345 is the approximate rate of change of life expectancy of U.S. males in 1910, in years. For 1950, we use the same technique, averaging slopes of two secant lines.

$$\frac{E(1960)-E(1950)}{10} \quad \text{and} \quad \frac{E(1950)-E(1940)}{10}.$$

Averaging these gives us .195 years is the approximate rate of change of life expectancy of U.S. males at $t = 1950$.

3. Computing limits.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.
Section 2.3 #10 Answer: 3/5
Section 2.3 #12 Answer: 3/2
Section 2.3 #16 Answer: 32
Section 2.3 #18 Answer: −1/9

4.

a. If $\lim_{x\to\infty} \frac{\ln f(x)}{x} = \alpha$, then $\lim_{x\to\infty} e^{\frac{\ln f(x)}{x}} = e^{\alpha}$.

But,

$$e^{\frac{\ln f(x)}{x}} = e^{\frac{1}{x}\ln f(x)} = e^{\ln f(x)^{1/x}} = f(x)^{1/x}.$$

Thus,

$$\lim_{x\to\infty} f(x)^{1/x} = e^{\alpha} = \beta.$$

b. If $d^x \le f(x) \le 2d(2d-1)^{x-1}$, then $d \le f(x)^{1/x} \le (2d(2d-1)^{x-1})^{1/x}$. By the theorem on page 116,

$$\lim_{x\to\infty} d \le \lim_{x\to\infty} f(x)^{1/x} \le \lim_{x\to\infty} (2d(2d-1)^{x-1})^{1/x}.$$

We consider each term separately:

(i) $\lim_{x\to\infty} d = d$ by law #7 on page 112.

(ii) $\lim_{x\to\infty} f(x)^{1/x} = \beta$ by (a).

(iii) Here is the tricky part: $2d(2d-1)^{x-1} = \frac{2d(2d-1)^x}{2d-1} = \frac{2d}{2d-1}(2d-1)^x$. Therefore,

$$(2d(2d-1)^{x-1})^{1/x} = \left(\frac{2d}{2d-1}\right)^{1/x}(2d-1).$$

Thus,

$$\lim_{x\to\infty} (2d(2d-1)^{x-1})^{1/x} = \lim_{x\to\infty} \left(\frac{2d}{2d-1}\right)^{1/x}(2d-1) = (2d-1)\lim_{x\to\infty}\left(\frac{2d}{2d-1}\right)^{1/x}$$

by law #3.

Since $d \ge 1$, by graphing or using a table, we see that

$$\lim_{x\to\infty}\left(\frac{2d}{2d-1}\right)^{1/x} = 1.$$

Thus,

$$\lim_{x\to\infty} (2d(2d-1)^{x-1})^{1/x} = 2d-1.$$

And finally, we combine (i), (ii), and (iii) to conclude that

$$d \le \beta \le 2d-1.$$