Math 111.17 Fall 2002
Assignment #10 Solutions

Problem 3

To show $\lim_{x\to 0}(\sec x)^{1/x^2} = \sqrt{e}$ we proceed as follows:

$$\begin{aligned}\lim_{x\to 0}(\sec x)^{1/x^2} &= \lim_{x\to 0} e^{\ln(\sec x)^{1/x^2}}\\ &= e^{\lim_{x\to 0}\ln(\sec x)^{1/x^2}}\\ &= e^{\lim_{x\to 0}\frac{1}{x^2}\ln(\sec x)}\\ &= e^{\lim_{x\to 0}\frac{\ln(\sec x)}{x^2}}.\end{aligned}$$

Now, we must compute $\lim_{x\to 0}\frac{\ln(\sec x)}{x^2}$ which is indeterminant $\left[\frac{0}{0}\right]$, so that we can use L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{\ln(\sec x)}{x^2} \overset{H}{=} \lim_{x\to 0}\frac{\frac{\sec x\tan x}{\sec x}}{2x} = \lim_{x\to 0}\frac{\tan x}{2x} \overset{H}{=} \lim_{x\to 0}\frac{\sec^2 x}{2} = \frac{1}{2}.$$

Hence,

$$\lim_{x\to 0}(\sec x)^{1/x^2} = e^{1/2} = \sqrt{e}.$$

Problem 4

To compute $\lim_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)}$ which is indeterminant $[1^\infty]$, we proceed as above.

$$\begin{aligned}\lim_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)} &= \lim_{x\to a} e^{\ln\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)}}\\ &= e^{\lim_{x\to a}\ln\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)}}\\ &= e^{\lim_{x\to a}\frac{1}{(x-a)}\ln\left(\frac{\sin x}{\sin a}\right)}\\ &= e^{\lim_{x\to a}\frac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a}}\end{aligned}$$

Now, we must compute $\lim_{x\to a}\frac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a}$ which is indeterminant $\left[\frac{0}{0}\right]$, so that we can use L'Hôpital's Rule.

$$\lim_{x\to a}\frac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a} = \lim_{x\to a}\frac{\ln(\sin x)-\ln(\sin a)}{x-a} \overset{H}{=} \lim_{x\to a}\frac{\frac{\cos x}{\sin x}}{1} = \frac{\cos a}{\sin a}$$

since $\sin a \neq 0$.

Hence,

$$\lim_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)} = e^{\frac{\cos a}{\sin a}}.$$