Math 105 Prelim #3 – Solutions

1.

(a) The mode is the most frequently occuring data point, which is 38. (It occurs 18 times.)

(b) The median is the middle number when the data are ordered in increasing rank. In this case, that is the 34th ranked number; thus the median is 37.

(c) The mean is given by

$$\overline{x} = \frac{2 \times 33 + 2 \times 34 + 9 \times 35 + 11 \times 36 + 12 \times 37 + 18 \times 38 + 13 \times 39}{67} \approx 37.015.$$

(d) The variance is given by

$$s^2 \approx \frac{(2 \times 33^2 + 2 \times 34^2 + 9 \times 35^2 + 11 \times 36^2 + 12 \times 37^2 + 18 \times 38^2 + 13 \times 39^2) - 67(37.015)^2}{67 - 1}$$

so that the standard deviation is

$$s = +\sqrt{s^2} \approx 1.59.$$

2.

(a) Let X be a normally distributed random variable with mean $\mu = 16$ and standard deviation $\sigma = 0.5$ which represents a randomly chosen bag of grapes. Thus,

$$P(X > 17) = P\left(\frac{X - 16}{0.5} > \frac{17 - 16}{0.5}\right) = P(Z > 2)$$

where $Z = (X - 16)/0.5$ is a normal random variable with mean 0 and standard deviation 1. From Table 1, $P(Z > 2) = 1 - P(Z \leq 2) = 1 - 0.9772 = 0.0228$.

(b) The event that at least one will weigh more than 17 ounces is the complement of the event that none will weigh more than 17 ounces. Therefore,

$$\begin{aligned} P(\text{at least one of these 3 weighs more than 17}) &= 1 - P(\text{none of these 3 weigh more than 17}) \\ &= 1 - \binom{3}{0}(0.0228)^0(0.9772)^3 \\ &\approx 0.06685. \end{aligned}$$

(c) If we look in Table 1, we can easily see that if Z is a normal random variable with mean 0 and standard deviation 1, then $P(Z \leq -1.28) \approx 0.1$ or $P(Z > -1.28) \approx 0.9$. Since $Z = (X - 16)/0.5$ we find that

$$P(Z > -1.28) = P(X > 0.5(-1.28) + 16) = P(X > 15.36) \approx 0.9.$$

Thus roughly 90% of the grapes are heavier than 15.36 ounces.

3.

(a) This is binomial with probability of success $p = 0.9$ and number of trials $n = 10$.

$$P(\text{exactly 9 receive letter on Tuesday}) = \binom{10}{9}(0.9)^9(0.1)^1 \approx 0.3874.$$

(b) For any binomial random variable X, we have $E(X) = np$. Thus we expect $np = 10 \times 0.9 = 9$ of our friends to receive the letter on Tuesday.

(c)

$$\begin{aligned} &P(\text{less than 8 receive letter on Tuesday}) = 1 - P(\text{at least 8 receive letter on Tuesday}) \\ &\quad = 1 - \left[\binom{10}{8}(0.9)^8(0.1)^2 + \binom{10}{9}(0.9)^9(0.1)^1 + \binom{10}{10}(0.9)^{10}(0.1)^0\right] \\ &\quad \approx 0.0702. \end{aligned}$$

4.

(i) Since

$$p = P(\text{3 ones}) = \binom{6}{3}(1/6)^3(5/6)^3$$

and

$$q = P(\text{4 twos}) = \binom{6}{4}(1/6)^4(5/6)^2$$

we can check that $p > q$.

(ii) Five cards are drawn from a standard deck of cards (with replacement): Since

$$p = P(> 3 \text{ face cards}) = \binom{5}{4}(12/52)^4(40/52)^1 + \binom{5}{5}(12/52)^5(40/52)^0$$

and

$$q = P(< 2 \text{ red cards}) = \binom{5}{0}(26/52)^0(26/52)^5 + \binom{5}{1}(26/52)^1(26/52)^4$$

we can check that $q > p$.

(iii) Note that $q = P(10 - n \text{ heads}) = P(n \text{ tails})$, and that by symmetry $P(n \text{ heads}) = P(n \text{ tails})$ so that $p = q$.

5.

(a) There are $10 \times 9 \times 8 \times 7 = 5040$ possible four digit numbers that you are allowed to pick. Thus, the probability of winning a prize of \$500 is

$$\frac{1}{5040}.$$

(b) Your expected profit is your expected winnings less your expected cost. Thus, your expected profit is

$$\left(500 \times \frac{1}{5040} + 0 \times \frac{5039}{5040}\right) - 2 \approx -1.90.$$

In other words, an expected profit of -\$1.90 means you expect a net loss of \$1.90 when playing this lottery.

(c) This game is clearly unfair since you do not expect to break even when playing it.

(d) Let X be the unknown prize amount. If the game were fair, then X would satisfy

$$\left(X \times \frac{1}{5040} + 0 \times \frac{5039}{5040}\right) - 2 = 0.$$

Thus, $X = 10080$, or the prize amount should be \$10,080 in order for the lottery to be fair.

6.

Let E be the event {mutated DNA} and let F be the event {4 of 7 septuplets gain superpowers}. Then the problem asks us to determine $P(E|F)$, which can be computed via Bayes' formula:

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E')P(E')}.$$

We are told that $P(E) = 0.01$ so that $P(E') = 0.99$.

Given E, there is a 10% chance that someone gains superpowers so that the probability that 4 of 7 gain superpowers is

$$P(F|E) = \binom{7}{4}(0.10)^4(0.90)^3 = 0.0025515.$$

However, given E', there is only a 4% chance that someone gains superpowers so that the probability that 4 of 7 gain superpowers is

$$P(F|E') = \binom{7}{4}(0.04)^4(0.96)^3 \approx 0.00007927.$$

Combining all of this gives

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E')P(E')} \approx \frac{0.0025515 \times 0.01}{0.0025515 \times 0.01 + 0.00007927 \times 0.99} \approx 0.245.$$

7.

(a) Since Y is binomial with probability of success $p = 2/6$ and number of trials $n = 98$, we have

$$E(Y) = np = \frac{98}{3}.$$

(b) Since Y has standard deviation $\sqrt{np(1-p)} = \sqrt{98 \times 1/3 \times 2/3} = 14/3$, let X be a normal random variable with mean $\mu = 98/3$ and standard deviation $\sigma = 14/3$. Then, by the normal approximation to the binomial, we have

$$\begin{aligned} P(30 \leq Y \leq 40) &\approx P(29.5 \leq X \leq 40.5) \\ &= P(\frac{29.5 - 98/3}{14/3} \leq \frac{X - 98/3}{14/3} \leq \frac{40.5 - 98/3}{14/3}) \\ &\approx P(-0.68 \leq Z \leq 1.68) \\ &\approx 0.9535 - 0.2483 = 0.7052 \end{aligned}$$

where Z is a normal random variable with mean 0 and standard deviation 1, and the corresponding probabilities are found from Table 1.