**Math 105 Prelim #3 - Solutions to Practice Problems**

# Brief Solutions

Warning: Errors are possible!

**Problem 1:**
**(a)** 0.0222 **(b)** 0.1797 **(c)** 0.7766

**Problem 2:**
$-0.833$ dollars; not a fair game

**Problem 3:**
**(a)** 195 dollars
**(b)** $n = 3$ gives 180 dollars, $n = 4$ gives 214.08 dollars, $n = 5$ gives 217.06 dollars, $n = 6$ from part **(a)** was 195 dollars. To maximize expected revenue, use $n = 5$.

**Problem 4:**
mean = 250, standard deviation = $5\sqrt{30}/2$, $P(-0.77 < z < 1.5) = 0.7126$

**Problem 5:**
**(a)** mean for frequency distribution 9.3; mean for probability distribution 5
**(b)** standard deviation for frequency distribution 3.9; standard deviation for probability distribution 1.58
**(c)** The interval is $1.84 \leq x \leq 8.16$.
**(d)** The histogram of the frequency distribution is not close enough to a normal curve.

**Problem 6:**
**(a)** Let $P'$ be probability in the 'IQ' normal distribution, and $P$ be probability in the standard normal distribution. Then $P'(x < 80) = P(z < (80 - \mu)/\sigma) = P(z < (80 - 100)/16) = P(z < -1.25) \simeq 0.1056$, where the actual probability is from the table for the standard normal distribution.

**(b)** $P'(80 < x < 120) = P((80 - 100)/16 < z < (120 - 100)/16) = P(-1.25 < z < 1.25) = P(z < 1.25) - P(z < -1.25) = (1 - P(z > 1.25)) - P(z < 1.25) = (1 - P(z < -1.25)) - P(z < -1.25) = 1 - 2P(z < -1.25) = 1 - 2(0.1056) \simeq 0.7888$. Note that, from our work in part **(a)**, no table is needed to solve this part.

**(c)** $P'(x > 140) = P(z > (140 - 100)/16) = P(z > 2.5) = 1 - P(z < 2.5) \simeq 1 - 0.9938 = 0.0062$.

**(d)** If we consider selecting a child with an IQ higher than 80 as a success, this is just a binomial probability with 5 trials, 4 successes, and probability of success $1 - 0.1056 = 0.8943$. So the probability is

$$\binom{5}{4}(0.8943)^4(0.1056) \simeq 0.3377.$$

**Problem 7:**
$P(-z^* < z < z^*) = 1 - 2P(z < -z^*)$. This can be seen clearly from a picture of the standard normal distribution. Now $P(-z^* < z < z^*) = .95 \Rightarrow 1 - 2P(z < -z^*) = .95 \Rightarrow 2P(z < -z^*) = .05 \Rightarrow P(z < -z^*) = .025$. Looking at a table, we see that $-z^*$ is $-2.81$, and so $z^* = 2.81$.

**Problem 8:**
**(a)** Suppose a die is rolled seven times. Count a roll as a success if it is either a 1 or a 3. Then this is the probability of exactly three successes out of the seven rolls.

**(b)** Suppose you have two coins and you flip them simultaneously. Call a flip a success if both coins show tails. Then this could be the probability of, in seven flips, the first three being successes and the remaining flips being failures.

**(c)** Suppose you have a bucket filled with four chalupas and five enchiladas. For dinner you select five items at random from the bucket. This is the probability of your dinner consisting of two chalupas and three enchiladas.

**Problem 9:**
**(a)** To get $X = 4$, there are two cases. First, you can roll a 2 on both dice. This probability is $2/6 \cdot 2/6 = 1/9$. The other case is that you roll a 3 on the first die and a 1 on the second. This probability is $4/6 \cdot 3/6 = 1/3$. So

$$P(X = 4) = \frac{1}{9} + \frac{1}{3} = \frac{4}{9}.$$

**b)** We need to find, for $i = 3, 5, 6$, $P(X = i)$. This is pretty much analogous to part **(a)**, so we just give the answers here: $P(X = 3) = \frac{1}{6}$, $P(X = 5) = \frac{5}{18}$, and $P(X = 6) = \frac{2}{18}$. Therefore,

$$E(X) = 3(\frac{1}{6}) + 4(\frac{4}{9}) + 5(\frac{5}{18}) + 6(\frac{2}{18}) = \frac{78}{18} = \frac{13}{3} \simeq 4.33.$$

**c)** This is clearly a binomial distribution, where rolling a 2 is considered a success. So $p = \frac{1}{3}$, and therefore $E(Y) = 72\frac{1}{3} = 24$.

**d)** We have $\mu = E(Y) = 24$, and $\sigma = \sqrt{np(1-p)} = \sqrt{72\frac{1}{3}\frac{2}{3}} = \sqrt{16} = 4$. So, if we let $P'$ and $P$ be as in problem 6, we have $P'(30 < Y < 40) \simeq P((30.5 - 24)/4 < z < (39.5 - 24)/4) = P(1.625 < z < 3.875) = P(z < 3.874) - P(z < 1.625)$. Since $z = 3.875$ isn't even on our table, we assume it is of probability 1. Therefore, the answer is $1 - P(z < 1.625) = 1 - 0.9484 = 0.0516$.

**Problem 10:**
**(a)** Write the values in order: 150, 180, 190, 230, 250, 250, 280, 300, 340, 380. The median is just the mean of the two middle numbers. Since these two numbers are both 250, the median is 250. The mean is a simple calculation: $(150 + 180 + 190 + 230 + 250 + 250 + 280 + 300 + 340 + 380)/10 = 255$. The standard deviation is calculated just as easily:

$$\sqrt{\frac{697300 - 10(255^2)}{9}} \simeq 72.$$

**(b)** The Bright Idea Lighting Company's bulbs have a higher mean life.

**(c)** For the Bright Idea Lighting Company, we have $P'(x > 350) = P(z > (350 - 262)/41) = P(z > 2.15) = 1 - P(z < 2.15) = 1 - 0.9842 = 0.0158$. For The Electric Company,

$$P'(z > 350) = P(z > \frac{350 - 255}{72}) = P(z > 1.32) = 1 - P(z < 1.32) = 1 - 0.9066 = 0.0934.$$

**Problem 11:**
**(a)** For the secret agent to live until Wednesday, he needs to live to Friday, then Saturday, then Sunday, then Monday, then Tuesday, then Wednesday. The probability of each of these happening is 0.49. Since we want the probability of all of them happening (i.e., their intersection), the answer is $(0.49)^6 \simeq 0.014$.

**(b)** The probability of any one of the secret agents being alive on Saturday is $(0.49)^2 \simeq 0.24$. This is a binomial distribution, so the expected number of successes (the secret agent living is considered a success) is just the probability of success times the number of trials (i.e., number of secret agents). So the answer is $(0.24)(12) = 2.88$.

**Problem 12:**
**(a)** $P(-r < z < r) = 0.9 \Rightarrow 1 - 2P(z < -r) = 0.9 \Rightarrow P(z < -r) = 0.05$. So, looking at the table we see that $-r \simeq -1.64$ (the actual value is between $-1.64$ and $-1.65$, but this is good enough). So the range is $-1.64 < x < 1.64$.

**(b)** $0.9 = P'(\mu - r < x < \mu + r) = P((\mu - r - \mu)/\sigma < z < (\mu + r - \mu)/\sigma) = P(-r < z < r)$. But we already found this number in part **(a)**, and so we know that $r = 1.64$. Therefore, the range is $10-1.64 < x < 10+1.64$, which is $8.36 < x < 11.64$.

**Problem 13:**
The probability of getting a sum greater than or equal to ten in one trial is

$$\frac{n(\{4,6\},\{6,4\},\{5,5\},\{5,6\},\{6,5\},\{6,6\})}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(where $S$ is the sample space of all possible pairs of rolls, and thus $n(S) = 36$). The answer is then found through binomial probability: $P$(fewer than nine of the outcomes are $\geq 10$) $= 1 - P$(nine outcomes are $\geq 10$) $- P$(ten outcomes are $\geq 10$) $=$

$$= 1 - \binom{10}{9}(\frac{1}{6})^9(\frac{5}{6})^1 - \binom{10}{10}(\frac{1}{6})^{10}(\frac{5}{6})^0 \simeq .9999992.$$

In other words, the probability is VERY high.

**Problem 14:**
Immediate binomial probability application:

$$\binom{5}{0}(0.7)^0(0.3)^5 + \binom{5}{1}(0.7)^1(0.3)^4 + \binom{5}{2}(0.7)^2(0.3)^3 \simeq 0.163.$$

**Problem 15:**
For all binomial probabilities in this problem, we consider the outcome 'heads' as a success. We also note that all probabilities take the form $C(\frac{1}{2})^m(\frac{1}{2})^{10-m} = C(\frac{1}{2})^{10}$, where $C$ is some binomial coefficient. Therefore, to consider the relation of $p$ to $q$, it is enough to simply consider the binomial coefficients, and ignore the $(\frac{1}{2})^{10}$ factor. For **(i)**, $p = \binom{10}{3} = 120$ and $q = \binom{10}{6} = 210$, so $p < q$. For **(ii)**, $p = 2^10 - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} - \binom{10}{3} = 848$ and $q = \binom{10}{9} + \binom{10}{10} = 10 + 1 = 11$, so $q > p$. For **(iiii)**, we simply note that the event "$n$ heads" is exactly the same as the event "$10 - n$ tails", and this gives $p = q$.

**Problem 16:**
**(i)** $P_N(x < \mu) = P(z < (\mu - \mu)/\sigma) = P(z < 0) = 0.5$.

**(ii)** $P_N(\mu - \sigma < x < \mu + \sigma) = P((\mu - \sigma - \mu)/\sigma < z < (\mu + \sigma - \mu)/\sigma) = P(-1 < z < 1) = P(z < 1) - P(z < -1) = 0.8413 - 0.1587 = 0.6826$.

**(iii)** $P_N(\mu - \frac{\sigma}{2} < x < \mu + \frac{\sigma}{2}) = P((\mu - \frac{\sigma}{2} - \mu)/\sigma < z < (\mu + \frac{\sigma}{2} - \mu)/\sigma) = P(-\frac{\sigma}{2}\frac{1}{\sigma} < z < \frac{\sigma}{2}\frac{1}{\sigma}) = P(-\frac{1}{2} < z < \frac{1}{2}) = P(z < \frac{1}{2}) - P(z < -\frac{1}{2}) = 0.6915 - 0.3085 = 0.383$.

**(iv)** $P_N(x > \mu + 2\sigma) = 1 - P_N(x < \mu + 2\sigma) = 1 - P(z < (\mu + 2\sigma - \mu)/\sigma) = 1 - P(z < 2) = 1 - 0.9772 = 0.0228$.

**Problem 17:**
There are many possible solutions here. A correct solution includes the following elements: if the monkeys were truly choosing the balls that random, then the probability that a given monkey selects the red ball is 1/3. Since there are 1000 monkeys, we expect that 1000/3, or roughly 333, of them will correctly select the red ball.

However, there were 435 that selected the red ball. Thus, the question becomes: how likely is observing 435 red balls from a binomial experiment of 1000 trials when the mean is 1000/3?

We can answer that question with the normal approximation. Suppose that $X$ is a normally distributed random variable with mean $\mu = np = 1000/3 \simeq 333.3$ and standard deviation $\sqrt{np(1-p)} = \sqrt{1000 \times 1/3 \times 2/3}$ $\simeq 14.91$. Then, the chance of observing at least 435 monkeys correctly picking the red ball *if* they were truly choosing randomly is

$$P(X \geq 435) = P(Z \geq \frac{435 - 333.3}{14.91}) = P(Z \geq 6.82).$$

Notice that the $z$-score 6.82 is not even on the table in the Appendix. This is becuase this probability is too close to zero to be registered. (Recall that 99.7% of normally distributed data lie within 3 standard deviations of the mean. Thus, less than 0.3% or 0.003 lie outside 3 standard deviations of the mean.)

From the TI-83 we can calculate that

$$P(Z \geq 6.82) \simeq 4.58 \times 10^{-12} = 0.00000000000458.$$

Thus, the chance of 435 monkeys randomly picking the red ball is extremely rare. Thus, there is sufficient evidence *based only on this data* that Prof. Frink's monkeys are intelligent. Hear Prof. Frink himself at:

`http://www.math.cornell.edu/~kozdron/Teaching/Cornell/105Fall03/frink.wav`