Math 105 Final Exam (Fall 2003) – Solutions

1.

(a) There is a fixed cost of \$50, and a variable cost of \$2 per guide. Let x be the number of guides sold. Therefore,

$$C(x) = 50 + 2x.$$

(b) Spencer's revenue is \$3 per guide sold. Therefore, $R(x) = 3x$. In order to break even, $R(x) = C(x)$ or $50 + 2x = 3x$. Solving for x we have

$$x = 50 \text{ guides}.$$

2.

(a) If E and F are independent, then $P(E \cap F) = P(E) \cdot P(F)$. If $P(E) = 0.4$ and $P(E \cap F) = 0.2$, this forces

$$P(F) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.4} = 0.5.$$

(b) Since $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, we have that

$$P(E \cup F) = 0.4 + 0.5 - 0.2 = 0.7.$$

3.

(a) If there are 40 tacos, then the given probabilities tell us that $n(J \cap G) = 4$, $n(J) = 12$, $n(G) = 10$. Since $n(J \cup G) = n(J) + n(G) - n(J \cap G)$, we have $n(J \cup G) = 12 + 10 - 4 = 18$. We are interested in $n((J \cup G)')$, the number that have neither jalapeño peppers nor green onions, which is $40 - 18 =$

$$22.$$

(b) We must compute $P(J'|G)$ which can be done by the defintion of conditional probability. Thus,

$$P(J'|G) = \frac{P(J' \cap G)}{P(G)} = \frac{1/4 - 1/10}{1/4} = \frac{3}{5}.$$

4.

We gather all of the coefficients of x, y, z, so that the given system is equivalent to

$$\begin{aligned} x + \quad y + \quad z &= 1 \\ -0.5x + 0.2y + 0.1z &= 0 \\ 0.1x - 0.7y + 0.2z &= 0 \\ 0.4x + 0.5y - 0.3z &= 0 \end{aligned}$$

which can be written as an augmented matrix as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.5 & 0.2 & 0.1 & 0 \\ 0.1 & -0.7 & 0.2 & 0 \\ 0.4 & 0.5 & -0.3 & 0 \end{array}\right].$$

$10R_2 \mapsto R_2$, $10R_3 \mapsto R_3$, and $10R_4 \mapsto R_4$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -5 & 2 & 1 & 0 \\ 1 & -7 & 2 & 0 \\ 4 & 5 & -3 & 0 \end{array}\right]$$

$5R_1 + R_2 \mapsto R_2$, $-R_1 + R_3 \mapsto R_3$, and $-4R_1 + R_4 \mapsto R_4$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 7 & 6 & 5 \\ 0 & -8 & 1 & -1 \\ 0 & 1 & -7 & -4 \end{array}\right]$$

$-7R_4 + R_2 \mapsto R_2$, $8R_4 + R_3 \mapsto R_3$, and $R_4 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -7 & -4 \\ 0 & 0 & -55 & -33 \\ 0 & 0 & 55 & 33 \end{array}\right]$$

$R_3 + R_4 \mapsto R_4$ and $-R_3/11 \mapsto R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -7 & -4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It is now easiest to back substitute. We find that $5z = 3$ or $z = 3/5$. Next, $y = -4 + 7z = -4 + 21/5 = 1/5$. Finally, $x = 1 - y - z = 1 - 1/5 - 3/5 = 1/5$. To summarize

$$x = \frac{1}{5}, \quad y = \frac{1}{5}, \quad z = \frac{3}{5}.$$

5.

(a) Let the order of the rows and columns be strawberry, chocolate, vanilla.

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}.$$

Since all of the entries of $P = P^1$ are non-zero, this Markov chain *is* regular.

(b) The equilibrium vector is found by solving $VP = V$ where $V = [v_1 \; v_2 \; v_3]$, and $v_1 + v_2 + v_3 = 1$. The corresponding system is therefore

$$\begin{aligned} v_1 + \quad v_2 + \quad v_3 &= 1, \\ 0.5v_1 + 0.2v_2 + 0.1v_3 &= v_1, \\ 0.1v_1 + 0.3v_2 + 0.2v_3 &= v_2, \\ 0.4v_1 + 0.5v_2 + 0.7v_3 &= v_3. \end{aligned}$$

However, note that this is exactly the system that was solved in Problem 4 (as per the hint!) with $x = v_1$, $y = v_2$, $z = v_3$. Hence,

$$V = [1/5 \;\; 1/5 \;\; 3/5].$$

(c) In order to determine the probability that Jay's grandchild will prefer vanilla ice-cream assuming Jay prefers chocolate ice-cream, we simply compute P^2 which is

$$P^2 = \begin{bmatrix} 0.31 & 0.16 & 0.53 \\ 0.21 & 0.21 & 0.58 \\ 0.16 & 0.21 & 0.63 \end{bmatrix}.$$

The required probability is the entry $(2,3)$ of P^2 which is

$$0.58.$$

6.

(a) In order to find the inverse of a matrix A, we must row reduce the augmented matrix $[A|I]$ to $[I|A^{-1}]$. Thus,

$$\left[\begin{array}{cc|cc} 0.4 & -0.2 & 1 & 0 \\ -0.1 & 0.3 & 0 & 1 \end{array}\right].$$

Now, $10R_1 \mapsto R_1$, $-10R_2 \mapsto R_2$, $R_1 \leftrightarrow R_2$ gives

$$\left[\begin{array}{cc|cc} 1 & -3 & 0 & -10 \\ 4 & -2 & 10 & 0 \end{array}\right].$$

Next, $-4R_1 + R_2 \mapsto R_2$ gives

$$\left[\begin{array}{cc|cc} 1 & -3 & 0 & -10 \\ 0 & 10 & 10 & 40 \end{array}\right].$$

Finally, $R_2/10 \mapsto R_2$ and $3R_2 + R_1 \mapsto R_1$ gives

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 4 \end{array}\right].$$

Thus, the required inverse is

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

(b) Fortunately, the matrix P is already decomposed into the blocks I, 0, R, Q. To determine the probabilities of absorption, we must compute

$$F = (I - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix},$$

using our answer to part **(a)**. We now compute the matrix product

$$FR = \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}.$$

Since we are interested in the probability of absorption in state A assuming the chain starts in state C, the required probability is entry $(1, 1)$ of the matrix FR which is

$$0.2.$$

7.

(a) Since there are 200 towns, and each town has 10 gin joints, there are 2000 possible gin joints for Ilsa to walk into. Since her choice is random, and each is equally likely, the required probability is

$$\frac{1}{2000}.$$

(b) This is binomial. The six girlfriends are hanging out in Rick's town, but each chooses independently, so there are 6 trials, where a "success" is a girlfriend selecting Rick's bar. Each trial therefore has success probability $p = 1/10$. Thus,

$$\begin{aligned} P(\text{fewer than } 3) &= P(\text{exactly } 0) + P(\text{exactly } 1) + P(\text{exactly } 2) \\ &= \binom{6}{0}(0.1)^0(0.9)^6 + \binom{6}{1}(0.1)^1(0.9)^5 + \binom{6}{2}(0.1)^2(0.9)^4. \end{aligned}$$

(c) Let X be a binomial random variable. We are interested in $P(X > 11)$. Note that we want to know the probability that *strictly more than 11* of Rick's ex-girlfriends visit his bar. Let Y be a normal random variable with mean $\mu = np = 100 \times 1/10 = 10$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{100 \times 1/10 \times 9/10} = 3$. Do not forget the "continuity correction" of 0.5. Thus,

$$\begin{aligned} P(X > 11) \approx P(Y \geq 11.5) = P\left(\frac{Y-10}{3} \geq \frac{11.5-10}{3}\right) &= P(Z \geq 0.5) \\ &\approx 1 - 0.6915 \\ &= 0.3085, \end{aligned}$$

where Z is a normal mean 0, SD 1 random variable, and $P(Z \geq 0.5)$ is computed from Table 1.

8.

(a) There are 20 objects that must be put into 12 distinguished positions. The number of ways this can be done is

$$P(20, 12) = \frac{20!}{(20-12)!} = \frac{20!}{8!}.$$

(b) In order to assign 8 people to lighting (2 people), sound (2 people), and props (4 people), we must compute the number of possible orderings of 8 things, and divide out the repetitions. Therefore, the number of possible assignments is

$$\frac{8!}{2!\,2!\,4!} = 420.$$

(c) There are 20 people auditioning for twelve positions. Since all the people auditioning have an equally likely chance of being selected, namely $12/20$, the chance that Mike is selected is therefore

$$\frac{12}{20}.$$

(d) There are $\binom{20}{12}$ ways to select 12 Angry Men from among the 20 auditioning. (Note this is asking the number of ways of casting 12 Angry Men without regard to order, which is a different question than in **(a)**.) Since 17 of those auditioning are women, there are $\binom{17}{12}$ ways to select 12 Angry Men from among these 17 auditioning. Therefore, the required probability is

$$\frac{\binom{17}{12}}{\binom{20}{12}} = \frac{14}{285}.$$

9.

(a) Since there are 10 coins, and each is tossed independently of the others, X is a binomial random variable with mean np, where $n = 10$ and $p = 1/2$. Thus,

$$E(X) = 10 \times \frac{1}{2} = 5.$$

(b) In order for $X = 3$, we must observe three heads. Since X is binomial, the required probability is

$$P(X = 3) = \binom{10}{3}(0.5)^3(0.5)^7 = \binom{10}{3} \cdot 2^{-10}.$$

(c) In order for $Y = 15$, we must have either (i) exactly 3 of 4 nickels show heads and all 6 dimes show tails, or (ii) exactly 1 of 4 nickels must show heads and exactly 1 of 6 dimes must show heads. Thus,

$$\begin{aligned} P(Y = 15) = {} & P(\text{3 of 4 nickels heads and 0 of 6 dimes heads}) \\ & + P(\text{1 of 4 nickels heads and 1 of 6 dimes heads}). \end{aligned}$$

However, since the nickels and dimes are tossed independently,

$$P(\text{3 nickels heads and 0 dimes heads}) = P(\text{3 nickels heads}) \cdot P(\text{0 dimes heads})$$

and

$$P(\text{1 nickel heads and 1 dime heads}) = P(\text{1 nickel heads}) \cdot P(\text{1 dime heads}).$$

Now, each of the corresponding probabilities is binomial. Therefore,

$$P(Y = 15) = \binom{4}{3}(0.5)^3(0.5)^1 \cdot \binom{6}{0}(0.5)^0(0.5)^6 + \binom{4}{1}(0.5)^1(0.5)^3 \cdot \binom{6}{1}(0.5)^1(0.5)^5.$$

(d) We must compute $P(Y = 15 | X = 3)$. By definition,

$$P(Y = 15 | X = 3) = \frac{P(X = 3 \text{ and } Y = 15)}{P(X = 3)}.$$

In order for both $X = 3$ and $Y = 15$ to occur, it must be the case that exactly 3 nickels show heads and no dimes show heads. However, this was computed in **(c)**, namely $\binom{4}{3}(0.5)^3(0.5)^1 \cdot \binom{6}{0}(0.5)^0(0.5)^6 = \binom{4}{3} \cdot \binom{6}{0} \cdot 2^{-10}$. We computed $P(X = 3)$ in **(b)**, so that

$$P(Y = 15 | X = 3) = \frac{\binom{4}{3} \cdot \binom{6}{0} \cdot 2^{-10}}{\binom{10}{3} \cdot 2^{-10}} = \frac{\binom{4}{3}}{\binom{10}{3}}.$$