Math 105 Exam 1 October 2, 2001 R. Durrett

[redacted]

(16 points) 1. A publisher sells math book for \$99. Each book costs \$34 to make but requires a fixed cost of \$1300 to set up the run (i.e., their cost function for x books is $1300 + 34x$). How many books do they need to sell to break even?

(16 points) 2. The Beta Alpha Delta fraternity has 93 members. 47 drink liquor, 72 drink beer, and 29 drink both. How many drink neither?

(20 points) 3. You roll two dice. In the game of craps, you win immediately in the event $A = \{$ the sum is 7 or 11 $\}$ and lose immediately in $B = \{$ the sum is 2, 3, or 12 $\}$. If neither of these happens, i.e., $C = A' \cap B'$, the game continues. Find the probabilities of the events A, B, and C.

(24 points, 16 for a, 8 for b) 4. (a) Find the inverse A^{-1} of the matrix $A = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$.
[A smart student will check that $AA^{-1} =$ the identity matrix.]
(b) Use the inverse you computed in (a) to solve

$$7x + 4y = 2$$
$$5x + 3y = 1$$

(20 points) 5. Xavier is eating at Friendly's at the Pyramid Mall with his two children Yoshi and Zev. Zev who is two years younger than Yoshi, says "Dad you are 20 years older than Yoshi and I combined, but if you add all of our ages you get 80." The waitress who is math major at Ithaca College writes on her order pad

$$x + y + z = 80$$
$$x - y - z = 20$$
$$y - z = 2$$

Find their ages. You may use either the echelon method or Gaussian elimination to solve the equations but show your work. (You do not have to say what row operations you are using.)

Math 105 **Answers to Exam 1. 10/2/2001** **R. Durrett**

1. In order for $1300 + 34x = 99x$, $6x = 1300$, or $x = 20$.

2. Let A be the event drinks liquor, B be drinks beer.

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 47 + 72 - 29 = 90$,

so $93 - 90 = 3$ drink neither.

3. sum	2	3	4	5	6	7	8	9	10	11	12
prob.	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$P(A) = 6/36 + 2/36 = 8/36$,

$P(B) = 1/36 + 2/36 + 1/36 = 4/36$,

$P(C) = 1 - P(A) - P(B) = (36 - 8 - 4)/36 = 24/36$.

4. (*a*)

$$\left.\begin{array}{cc} 7 & 4 \\ 5 & 3 \end{array}\right| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

$$\left.\begin{array}{cc} 1 & 4/7 \\ 0 & 1/7 \end{array}\right| \begin{array}{cc} 1/7 & 0 \\ -5/7 & 1 \end{array} \qquad \begin{array}{c} R1/7 \\ R2 - 5NR1 \end{array}$$

$$\left.\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right| \begin{array}{cc} 3 & -4 \\ -5 & 7 \end{array} \qquad \begin{array}{c} R1 - (4/7)NR2 \\ 7R2 \end{array}$$

(*b*)

$$\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 - 4 \\ -10 + 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

As in class $R1$ is short for row 1, $NR1$ for new row 1, etc.

5. Echeclon method.

$$x + y + z = 80$$

$$-2y - 2z = -60 \qquad E2 - E1$$

$$y - z = 2$$

$$x + y + z = 80$$

$$y + z = 30 \qquad E2/(-2)$$

$$-2z = -28 \qquad E3 - NE2$$

$$z = 14; \quad y + 14 = 30, \ y = 16$$

$$x + 16 + 14 = 80, \ x = 50$$

Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 80 \\ 1 & -1 & -1 & 20 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 80 \\ 0 & -2 & -2 & -60 \\ 0 & 1 & -1 & 2 \end{array}\right] \qquad \begin{array}{c} \\ R2 - R1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 30 \\ 0 & 0 & -2 & -28 \end{array}\right] \qquad \begin{array}{c} R1 - NR2 \\ R2/(-2) \\ R3 - NR2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 14 \end{array}\right] \qquad \begin{array}{c} \\ R2 - NR3 \\ R3/(-2) \end{array}$$

Math 105 Exam 2 **November 1, 2001** **R. Durrett**

[redacted] *With the exception of problem 5, you do not have to simplify your answers.*

1. (27 points) 28 fresh women live on one floor of one of the new North Campus dorms. How many way can they:

(a) be assigned to 28 beds (one person per bed!).

(b) choose a starting line-up for their hockey team: assume there are six positions to be filled all of which are different.

(c) choose a delegation of four people to go to complain to ~~Hunter Rawlings~~ Jeffrey Lehman about the cheap construction of the dorm, if one of the four is designated as the spokeswoman.

2. (12 points) Beavis has 11 books. Five are black, four are brown, and two are gray. If he puts these 11 books at random on a shelf, what is the probability that all of the books of the same color end up together? I.e., when we look at the shelf we see three groups of books each of which are all the same color.

3. (21 points) When Ryckie stays up late, he studies 30% of the time but 70% of the time he sneaks out in the backyard and smokes marijuana after his parents go to bed. 20% of the time that he studies he wakes up with bloodshot eyes, but that occurs 60% of the time when he smokes. Ryckie had bloodshot eyes this morning. What is the probability that he was studying late last night?

4. (12 points) Three dice are rolled. Let A be the event that the first and second show the same value. Let B be the event that the second and third show the same value. Are A and B independent? In addition to answering YES or NO give a reason for your answer.

5. (24 points) When I was teaching at UCLA, an elementary school teacher called up the math department and complained about a promotional game at McDonald's. They claimed you had a a 1/25 chance of winning a cash prize, but when she brought her 25 students to McDonald's, none of them won a prize! Calculate the probability that among the 25 students (a) 0 (b) 1 (c) 2 will win a prize.

[*Note: In order to achieve the full benefit from this exciting problem, I would like you to compute the probabilities in (a)–(c) to three decimal places.*]

1. (a) 28! (b) $P(28,6) = 28!/22! = 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23$

(c) First we choose the chairman then we choose the other three women:

$$28 \cdot C(27,3) = (28 \cdot 27 \cdot 26 \cdot 25)/3!$$

2. $(3!\,5!\,4!\,2!)/11!$. There are 11! possible orders for the books. To generate one with all three colors together we first decide on one of the 3! orders for the colors. Once that is done we can put down the black books in 5! ways, the brown ones in 4! ways, and the gray in 2! ways.

3. Let $A_1 =$ "studies," $A_2 =$ "smokes," $B =$ "bloodshot eyes." The definition of condition probability implies $P(A_1|B) = P(A_1 \cap B)/P(B)$. The multiplication rule implies

$$P(A_1 \cap B) = P(B|A_1)P(A_1) = 0.2 \cdot 0.3 = 0.06$$

$$P(A_2 \cap B) = P(B|A_2)P(A_2) = 0.6 \cdot 0.7 = 0.42$$

Combining the two we have $P(A_1|B) = 0.06/(0.06 + 0.42) = 1/8$.

4. $P(A) = P(B) = 6/36$, $P(A \cap B) = P($ all 3 are the same $) = 6/216$, so

$$P(A \cap B) = 1/36 = P(A) \cdot P(B)$$

YES the two events are independent.

5. (a) $(24/25)^{25} = 0.360$

(b) $C(25,1) \cdot (1/25) \cdot (24/25)^{24} = (24/25)^{24} = 0.375$

(c) $\frac{25 \cdot 24}{2} \cdot (1/25)^2 \cdot (24/25)^{23} = \frac{1}{2} \cdot (24/25)^{24} = 0.188$

Note: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$P(n,r) = \frac{n!}{(n-r)!}$

Math 105 Exam 3 **November 27, 2001** **R. Durrett**

[redacted]

Be very careful with your solution of part (a) of 2, since parts (b) and (c) depend on it.

1. (16 points) Find the sample mean, median, and sample standard deviation for the following data set: 16, 2, 35, 4, 8.

2. (32 points, 8 each part) Three balls are drawn WITHOUT replacement from an urn with 4 white and 6 blue balls. Let W be the total number of white balls drawn.

(a) Find $P(W = w)$ for $w = 0, 1, 2, 3$.

(b) Find the expected value, EW.

[redacted]

(d) Let N be the number of white balls when we draw three times from the same urn WITH replacement. Find the mean and variance of N.

3. (16 points) The lifetime of ACME car batteries has a normal distribution with mean 180 weeks and a standard deviation of 16 weeks. If the company guarantees the batteries for 3 years (156 weeks), what is the percentage of the batteries that will fail before the warranty is up?

4. (32 points, 16 each part) A pro football team plays a 16 game season. Suppose that the probability they win each game is 3/4 and the outcomes of different games are independent. Compute the probability that the team will win exactly 12 games (a) using the normal approximation, and (b) using the binomial distribution. *To obtain the full benefit from this fascinating problem you should work out the answer to (b) to four decimal places.*

Math 105 **Answers to Exam 3. 11/27/2001** **R. Durrett**

1. $\bar{X} = (2+4+8+16+35)/5 = 13$. The median is 8. $\Sigma(X^2) = 4+16+64+256+1125 = 1565$. $s^2 = (1565 - 5 \cdot (13)^2)/4 = 180$. $s = \sqrt{180} = 13.14$

2. (a) $P(W = w) = \binom{4}{w}\binom{6}{3-w}/\binom{10}{3}$. $\binom{10}{3} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$.

$$P(W=0) = \frac{\binom{6}{3}}{120} = \frac{20}{120} \qquad P(W=1) = \frac{\binom{4}{1}\binom{6}{2}}{120} = \frac{60}{120}$$
$$P(W=2) = \frac{\binom{4}{2}\binom{6}{1}}{120} = \frac{36}{120} \qquad P(W=3) = \frac{\binom{4}{3}}{120} = \frac{4}{120}$$

(b) The easy way to compute the expected value is to note $EW = EW_1 + EW_2 + EW_3$ where W_i is the number of white balls on the ith draw so $EW = 3(0.4) = 1.2$. One can also use the definition directly

$$EW = 1 \cdot \frac{60}{120} + 2 \cdot \frac{36}{120} + 3 \cdot \frac{4}{120} = \frac{60+72+12}{120} = \frac{144}{120} = 1.2$$

[redacted]

(d) N has a binomial distribution with $n = 3$ and $p = 0.4$ so $EN = np = 1.2$ and var $(N) = np(1-p) = 3(.4)(.6) = .72$. [redacted]

3. $$P(X \le 156) = P\left(\frac{X-180}{16} \le -\frac{24}{16}\right) = P(Z \le -1.33) = 0.0968$$

4. (a) The number of games they will win has mean $16 \cdot 3/4 = 12$ and standard deviation $\sqrt{16(3/4)(1/4)} = \sqrt{3} = 1.732$.

$$P(11.5 \le X \le 12.5) = P\left(\frac{-.5}{1.732} \le \frac{X-12}{1.732} \le \frac{-.5}{1.732}\right) = P(-.28 \le Z \le .28)$$
$$= P(Z \le 0.28) - P(Z \le -0.28) = 0.6141 - 0.3859 = 0.2282$$

(b) Binomial probability is

$$\binom{16}{12}(3/4)^{12}(1/4)^4 = \frac{16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{3^{12}}{4^{16}} = 1820 \cdot \frac{531441}{4294967296} = 0.2252$$

Math 105 **Final Exam. December 13, 2001** **R. Durrett**

[redacted] *You do not have to simplify your answer unless the problem tells you to.*

[redacted]

1. (50 points, 10 points each part) 15 students are taking a freshman seminar with Professor MADMANDA on the history of hemp.
(a) How many ways can they sit in the 20 chairs in the class room?
(b) How many ways can the professor give out 5 A's, 7 B's, and 3 C's?
(c) How many ways can the students rearrange the letters in his last name?
(d) If he puts the letters of his last name into a urn and draws out two, what is the probability the two letters will be the same?
(e) In the game of Scribble, M's are worth 4 points, D's 2 points, and N's and A's 1 point. What is the mean [redacted] of X, the point value of a letter chosen at random from his last name?

2. (20 points) In a math class 40% of the students got A's, 40% of the students got B's and 20% got C's. On the teacher evaluations, 90% of the A students said they were happy with how the course was taught, 70% of the B students were, but only 40% of the C students were. Fred put happy on the evaluation form. What is the probability he got an A? a B? a C?

3. (32 points $= 12 + 20$). In each course Maria has a 60% chance of getting an A and a 40% chance of getting a B. (a) What is the probability she will get 3 A's and 1 B this semester? Give the answer to four decimal places. (b) Use the normal approximation to the binomial to estimate the probability she will get 24 or more A's in her 32 courses at Cornell.

4. (32 points = 10+4+18). If the English language was a Markov chain then the transition probabilities between successive symbols in a document might be as follows

	V	C	S
Vowel	.1	.8	.1
Consonant	.5	.4	.1
Space	.6	.4	0

• 3-step transition from S to S.
• The (S,S) entry in P^3.

(a) Find the two step transition probability. (b) Find $p^3_{S,S}$. (c) Find the equilibrium frequencies of the three states.

5. (20 points). The equations $x - 3z = 4$, $x + 2y = 1$, $2y + 4z = -2$ can be written as

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}.$$ Find the inverse A^{-1} and use it to find x, y, z.

Math 105 **Answers to Final Exam. 12/13/2001** **R. Durrett**

1. (a) $P(20, 15) = 20!/5!$. Think of the students lining up in alphabetical order and choosing their seats one at a time.

(b) $C(15, 5) \cdot C(10, 7) = 15!/(5!7!3!)$. First he picks 5 of the 15 students to get an A then 7 of the 10 that remain to get a B.

(c) $8!/(2!2!3!)$

(d) $\left[\binom{2}{2} + \binom{2}{2} + \binom{3}{2}\right] / \binom{8}{2} = 5/28$

(e) $EX = 4 \cdot (2/8) + 2 \cdot (2/8) + 1 \cdot (4/8) = 2.$

$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$P(n,r) = \frac{n!}{(n-r)!}$

[redacted]

2. Let H denote happy and A, B, C the possible grades. The definition of conditional probability implies $P(A|H) = P(A \cap H)/P(H)$. Using the multiplication rule three times

$$P(A \cap H) = P(A)P(H|A) = 0.4 \cdot 0.9 = 0.36$$
$$P(B \cap H) = P(B)P(H|B) = 0.4 \cdot 0.7 = 0.28$$
$$P(C \cap H) = P(C)P(H|C) = 0.2 \cdot 0.4 = 0.08$$

From this we see $P(H) = 0.36 + 0.28 + 0.08 = 0.72$,

$$P(A|H) = \frac{0.36}{0.72} = \frac{1}{2} \quad P(B|H) = \frac{0.28}{0.72} = \frac{7}{18} \quad P(C|H) = \frac{0.08}{0.72} = \frac{1}{9}$$

3. (a) Using the binomial distribution $\binom{4}{3}(.6)^3(.4) = 0.3456$

(b) Let X be the number of A's. $EX = 32(0.6) = 19.2$, the standard deviation of X is $\sqrt{32(0.60(0.4)} = 2.77$.

$$P(X \geq 23.5) = P\left(\frac{X - 19.2}{2.77} \geq \frac{4.3}{2.77}\right) = P(Z \geq 1.55)$$
$$= 1 - P(Z \leq 1.55) = 1 - 0.9394 = 0.0606$$

4. (a)

$$\begin{pmatrix} .1 & .8 & .1 \\ .5 & .4 & .1 \\ .6 & .4 & 0 \end{pmatrix} \begin{pmatrix} .1 & .8 & .1 \\ .5 & .4 & .1 \\ .6 & .4 & 0 \end{pmatrix} = \begin{pmatrix} .47 & .44 & .09 \\ .31 & .60 & .09 \\ .26 & .64 & .10 \end{pmatrix}$$

(b) $p^3_{ss} = .26(.1) + .64(.1) = .09$

(c) The equations we want to solve to find the stationary distribution are

$$
\begin{aligned}
x + y + z &= 1 \\
.1x + .5y + .6z &= x \\
.8x + .4y + .4z &= y \\
.1x + .1y &= z
\end{aligned}
$$

Using the last equation in the first two we have

$$1.1x + 1.1y = 1 \qquad -.84x + .56y = 0$$

The second equation implies $y = (.84x)/.56 = 3x/2$. Using this in the first one we have $1.1(5x/2) = 1$ or $x = 4/11$. From this we get $y = 3x/2 = 6/11$, $z = 1 - x - y = 1/11$.

5. To find the inverse

$$
\left.\begin{array}{ccc|ccc}
1 & 0 & -3 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 \\
0 & 2 & 4 & 0 & 0 & 1
\end{array}\right.
\quad\rightarrow\quad
\left.\begin{array}{ccc|ccc}
1 & 0 & -3 & 1 & 0 & 0 \\
0 & 2 & 3 & -1 & 1 & 0 \\
0 & 2 & 4 & 0 & 0 & 1
\end{array}\right.
\quad\rightarrow
$$

$$
\left.\begin{array}{ccc|ccc}
1 & 0 & -3 & 1 & 0 & 0 \\
0 & 1 & 3/2 & -1/2 & 1/2 & 0 \\
0 & 0 & 1 & 1 & -1 & 1
\end{array}\right.
\quad\rightarrow\quad
\left.\begin{array}{ccc|ccc}
1 & 0 & 0 & 4 & -3 & 3 \\
0 & 1 & 0 & -2 & 2 & -3/2 \\
0 & 0 & 1 & 1 & -1 & 1
\end{array}\right.
$$

Using the inverse we can now solve the equations by

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & 3 \\ -2 & 2 & -3/2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix}
$$

That is, $x = 7$, $y = -3$, $z = 1$.