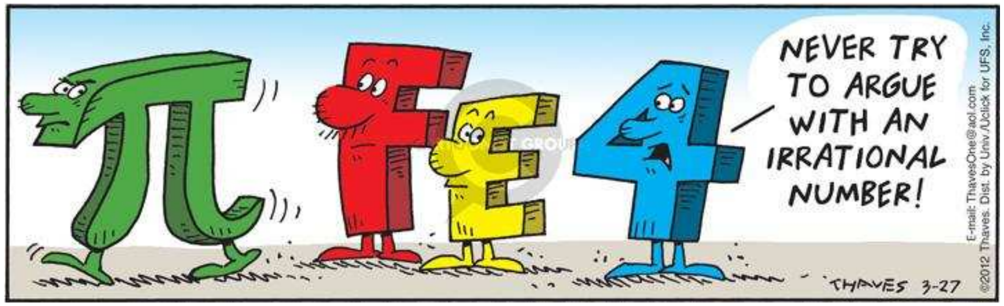
π-Day, 2013



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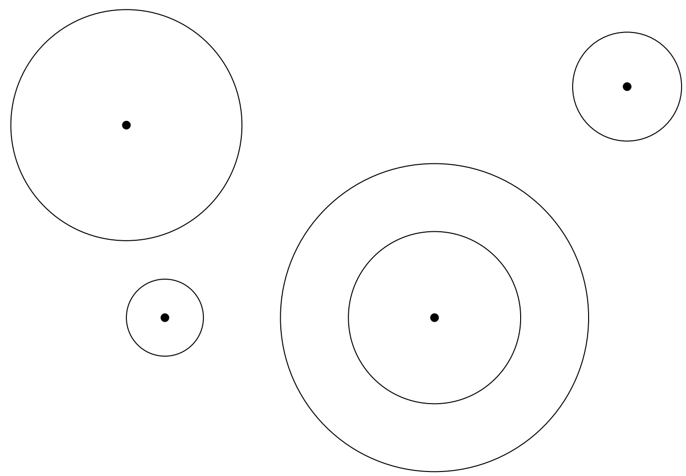
What is π?

In any circle, the ratio of the circumference to the diameter is constant.

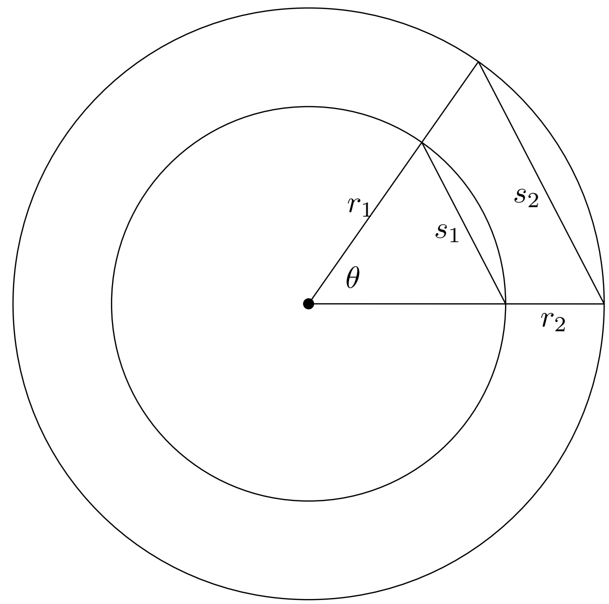
We are taught in high school that this number is called π. That is,

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

for any circle.



But how do you know that the ratio is constant?
Have you ever thought of proving this?
Could there exist two circles having different ratios of circumference to diameter?



Partition the circles into $n \geq 3$ congruent wedges, each with angle $\theta = 360^\circ/n$.

Note that the two triangles T_1 and T_2 are similar. Therefore, $\frac{s_1}{r_1} = \frac{s_2}{r_2}$.

Moreover, $C_1 \approx ns_1$ and $C_2 \approx ns_2$. In fact, $C_1 = \lim_{n\to\infty} ns_1$ and $C_2 = \lim_{n\to\infty} ns_2$.

Therefore,

$$\frac{C_1}{d_1} = \frac{C_1}{2r_1} = \frac{\lim_{n\to\infty} ns_1}{2r_1} = \lim_{n\to\infty} \frac{ns_1}{2r_1} = \lim_{n\to\infty} \frac{ns_2}{2r_2} = \frac{\lim_{n\to\infty} ns_2}{2r_2} = \frac{C_2}{2r_2} = \frac{C_2}{d_2}$$

So now that we know π exists, can we calculate it?

It seems that the symbol π for the ratio of the circumference to the diameter of a circle was first used by the Welsh mathematician William Jones in 1706.

Leonard Euler adopted it in 1737 and it quickly became the standard.

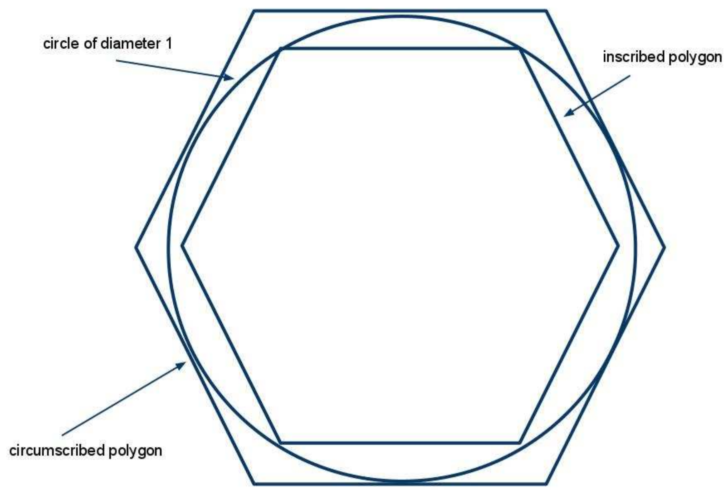
However, the fact that the ratio is constant has been known for about 3600 years!

Hence, calculating the value of π has interested people for millennia.

The first to calculate π theoretically appears to be Archimedes of Syracuse (287–212 BC).

By inscribing and circumscribing a 96-sided polygon, he computed

$$\frac{223}{71} < \pi < \frac{22}{7}.$$



For the next 1800 years, the only improvements in approximations to π came from geometry and using polygons with more and more sides.

However, the European Renaissance (c. 1600) brought about a whole new mathematical world including calculus.

Gottfried Wilhelm von Leibniz (1646–1716) and Isaac Newton (1642–1727) are both recognized as inventing calculus independently.

One remarkable result from calculus that proved useful for approximating π for over 200 years is known as Leibniz's formula—although it may have been discovered first by James Gregory (1638–1675).

Leibniz-Gregory Formula for π

Start with the geometric series formula that we learn in high school

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

which is valid for $|r| < 1$. If we substitute in $r = -t^2$, then

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

and so integrating from 0 to x yields

$$\int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right)\,\mathrm{d}t$$

which implies

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots .$$

It can be checked that this formula is valid for $|x| \le 1$.

Leibniz-Gregory Formula for π

Since $\tan(\pi/4) = 1$, if we substitute $x = 1$ into

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$

we find

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots .$$

That is,

$$\boxed{\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right)}$$

and so to approximate π, one just needs to add up a lot of terms in this series.

Notice that successive terms in this alternating series are reciprocals of odd integers. This means that the convergence will be very slow!

Hence, one will need to add up a LOT of terms to get even a few digits accuracy.

In fact, one needs to add up the first 300 terms to get the two decimal place accuracy of Archimedes' $\frac{22}{7}$.

Improved Leibniz-Gregory Formulas for π

Although the original Leibniz-Gregory formula converges very slowly to π, with a clever application of a high school trig identity we can do a lot better.

Recall the angle-sum formula for tangent, namely

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}. \tag{$*$}$$

Suppose that $\alpha = \arctan(1/2)$ and $\beta = \arctan(1/3)$. If we let $\theta = \alpha + \beta$, and observe that both α and β are acute angles, then $\theta < \pi$ implying that θ is in either the first or second quadrant.

From $(*)$ we find

$$\tan(\theta) = \tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = 1.$$

Thus, if $\tan(\theta) = 1$ and θ is in either the first or second quadrant, we conclude that $\theta = \arctan(1) = \frac{\pi}{4}$; that is,

$$\theta = \alpha + \beta$$

or, equivalently,

$$\boxed{\frac{\pi}{4} = \arctan(1/2) + \arctan(1/3).}$$

Thus, if we substitute $1/2$ and $1/3$ into our formula for $\arctan$, we obtain

$$\begin{aligned}\frac{\pi}{4} = &\left((1/2) - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} + \frac{(1/2)^9}{9} + \cdots\right) \\ &+ \left((1/3) - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} + \frac{(1/3)^9}{9} + \cdots\right)\end{aligned}$$

Machin's Formula

Using the same trick with the tangent formula, one can show

$$\boxed{\frac{\pi}{4} = 4\arctan(1/5) + \arctan(1/239).}$$

This formula was discovered by John Machin and he used it to calculate π to 100 decimals in 1706.

In 1798, Euler found

$$\boxed{\pi = 20\arctan(1/7) + 8\arctan(3/79).}$$

Here's an cute formula for you to prove.

$$\boxed{\pi = \arctan(1) + \arctan(2) + \arctan(3).}$$

Machin's Method (1706–1949)

Using Machin's method, William Shanks determined π to 607 places in 1853 and 707 places in 1873.

However, it turns out that Shanks made a mistake and was only accurate to 527 places.

Curiously, it was noticed by the mathematician Augustus De Morgan around 1873 that there was a suspicious shortage of 7s in the last few hundred digits in Shanks' expansion.

Then in 1946, D.F. Ferguson found that Shanks made an error.

But by 1949, calculating π by hand was obsolete as computers were now being used to do the job.

Here is π to roughly 1500 digits.

3.1415926535897932384626433832795028841971693993751058209749445923078164
062862089986280348253421170679821480865132823066470938446095505822317253
594081284811174502841027019385211055596446229489549303819644288109756659
334461284756482337867831652712019091456485669234603486104543266482133936
072602491412737245870066063155881748815209209628292540917153643678925903
600113305305488204665213841469519415116094330572703657595919530921861173
819326117931051185480744623799627495673518857527248912279381830119491298
336733624406566430860213949463952247371907021798609437027705392171762931
767523846748184676694051320005681271452635608277857713427577896091736371
787214684409012249534301465495853710507922796892589235420199561121290219
608640344181598136297747713099605187072113499999983729780499510597317328
160963185950244594553469083026425223082533446850352619311881710100031378
387528865875332083814206171776691473035982534904287554687311595628638823
537875937519577818577805321712268066130019278766111959092164201989380952
572010654858632788659361533818279682303019520353018529689957736225994138
912497217752834791315155748572424541506959508295331168617278558890750983
817546374649393192550604009277016711390098488240128583616035637076601047
101819429555961989467678374494482553797747268471040475346462080466842590
694912933136770289891521047521620569660240580381501935112533824300355876
402474964732639141992726042699227967823547816360093417216412199245863150
302861829745557067498385054945885869269956909272107975093029553211653449

π *today*

Currently, the “world record” for digits of π is over 10 trillion digits.

So why do people want to compute π to so many digits?

Superficial answer: it’s fun!

Practical answer: it can be used for testing supercomputers and numerical analysis algorithms.

Calculating the digits to millions of decimal places is now used to test computers for bugs in hardware and software (which is how Intel’s Pentium found a chip bug a few years ago).

Modern record breakers still add up terms in infinite series known to converge to π.

The more rapidly converging the series, the “easier” it is to get more digits in the approximation.

Actually, the real breakthrough is in the fact that these recently-discovered algorithms allow digits of π to be calculated without requiring the computation of earlier digits.

The Bailey-Borwein-Plouffe formula (BBP formula) provides such a digit extraction algorithm for the computation of the nth binary digit of π using base 16 math. (For details, see Wikipedia.)

This summation formula was discovered in 1995 by Simon Plouffe and is named after the authors of the paper in which the formula was published, David H. Bailey, Peter Borwein, and Simon Plouffe.

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$