A Random Look at Brownian Motion

Michael Kozdron
Duke University

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http://www.math.cornell.edu/~kozdron/

Some Early History

In the summer of 1827, the Scottish botanist Robert Brown observed that microscopic pollen grains suspended in water move in an erratic, highly irregular, zigzag pattern.

Following Brown's initial report, other scientists verified the strange phenomenon. Brownian motion was apparent whenever very small particles were suspended in a fluid medium, for example smoke particles in air.

It was eventually determined that finer particles move more rapidly, that their motion is stimulated by heat, and that the movement is more active when the fluid viscosity is reduced.

It was only in 1905 that Einstein, using a probabilistic model, could provide a satisfactory explanation.

He asserted that the Brownian motion originates in the continual bombardment of the pollen grains by the molecules of the surrounding water, with successive molecular impacts coming from different directions and contributing different impulses to the particles. As a result of the continual collisions, the particles themselves had the same average kinetic energy as the molecules.

Note that in 1905, belief in atoms and molecules was far from universal.

In 1905, Einstein also published seminal papers on the special theory of relativity, and the photoelectric effect (for which he won a Nobel prize).

Naturally their paths must be continuous, but they were seen to be so irregular that the French physicist Jean Perrin believed them to be non-differentiable. (Weierstrass had recently discovered such pathological functions do exist. Indeed the continuous function

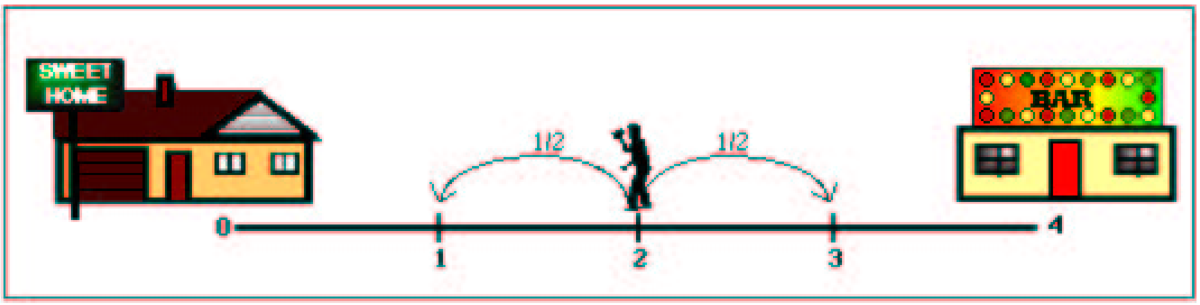
$$g(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

where a is odd, $b \in (0,1)$, and $ab > 1 + 3\pi/2$ is nowhere differentiable.) Perrin himself worked to show that colliding particles obey the gas laws, calculated Avogadro's number, and won the 1926 Nobel prize.

In 1923, the mathematician Norbert Weiner established the mathematical existence of Brownian motion. That is, he verified the existence of a stochastic process with the given properties.

(Simple) Random Walk on $\mathbb{Z}$

Suppose that we are at the origin and we perform the following motion. Flip a fair coin. If it lands heads, take a step right. If it lands tails, take a step left. Continue.



This is a one dimensional random walk on the integers.

If we let X_i denote the outcome of the i^{th} flip, then we write

$$\mathbb{P}\{X_i = +1\} = \mathbb{P}\{X_i = -1\} = 1/2.$$

Denote by S_n our position at the n^{th} step, so that

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Note that for each n, our position S_n is random. Thus, we call S_n a random variable, and the collection of random variables $\{S_n : n = 0, 1, \ldots\}$, a stochastic process.

We call the process $\{S_n\}$ a *(simple) random walk* on $\mathbb{Z}$ starting at the origin.

Since each flip can be either $+1$ or -1, and each occurs with equal probability, we expect the average, or mean, flip to be 0. We write, $\mathbb{E}(X_i) = 0$.

Since S_n is the sum of n mean-zero flips, we expect $\mathbb{E}(S_n) = 0$ for each n.

Question. What is our expected displacement after n steps, namely $\mathbb{E}(|S_n|)$?

Notice that $X_i^2 = 1$, so that $\mathbb{E}(X_i^2) = 1$.

In a sense, this tells us that our expected squared displacement each step is 1, so that after n steps, our expected squared displacement should be n.

It is, in fact, true (and easy to show) that $\mathbb{E}(S_n^2) = n$.

This suggests that $\mathbb{E}(|S_n|) \approx \sqrt{n}$.

Normal Distribution

The formula

$$n(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

describes a "bell-curve" centred at μ with variance σ^2 (or spread σ).

A random variable N is normally distributed with mean μ and variance σ^2, written $\mathcal{N}(\mu, \sigma^2)$, if N has this density.

Picture

That is, if

$$\mathbb{P}\{N \le x\} = \int_{-\infty}^{x} n(y)\, dy = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{(y-\mu)^2}{2\sigma^2})\, dy.$$

Central Limit Theorem

One remarkable result of elementary probability is the Central Limit Theorem.

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} = \frac{S_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0,1)$$

That is, the distribution of our random walk, normalized by $\sqrt{n}$, converges to the distribution of a normal random variable.

If $A \subseteq \mathbb{R}$ open interval, then

$$\lim_{n\to\infty} \mathbb{P}\{\frac{S_n}{\sqrt{n}} \in A\} = \frac{1}{\sqrt{2\pi}} \int_A \exp(-\frac{y^2}{2})\, dy.$$

[This is sometimes referred to as the binomial approximation to the normal.]

Picture

Brownian Motion

A one-dimensional real-valued stochastic process $\{B_t, t \geq 0\}$ is a Brownian motion if

- $B_0 = 0$ and the function $t \mapsto B_t$ is continuous (with probability one),
- for any $t_0 < t_1 < \ldots < t_n$ the increments $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_{n-1}} - B_{t_n}$ are independent
- for any s, $t \geq 0$, the increment $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ is normally distributed.

Graphing Random Walks

Let $\omega = (X_1, X_2, X_3, \ldots)$ be the result of a sequence of coin flips. Then, we can plot (n, S_n) and visualize the path of our random walk.

Convergence of Random Walks?

We can turn our RW into a continuous function on $[0,1]$ as follows.

Fix an integer n. Instead of having time increments of size 1, we will take a step at every $1/n$ unit of time.

Question. How far should each step be?

Let us for now take steps of size a_n.

For $j = 0, 1, \ldots, n$, let

$$f_n(j/n) = a_n S_j.$$

Let us choose the step size a_n so that $f_n(1) = f_n(n/n) = a_n S_n$ has variance 1.

By definition, $\mathrm{var}(Y) := \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2$.

Since $\mathbb{E}(S_n) = 0$, we conclude that

$$\mathbb{E}(f_n(1)^2) = \mathbb{E}(a_n^2\, S_n^2) = a_n^2 \cdot n = 1 \text{ if } a_n = \frac{1}{\sqrt{n}}.$$

Consequently, for $j = 0, 1, \ldots, n$, let

$$f_n(j/n) = \frac{1}{\sqrt{n}} S_j.$$

To define f_n for other t, linearly interpolate.

Thus,

$$f_n(t) = \frac{1}{\sqrt{n}} \left[S_{[\![nt]\!]} + (nt - [\![nt]\!]) \cdot \left(S_{[\![nt]\!]+1} - S_{[\![nt]\!]} \right) \right]$$

where $[\![x]\!]$ is the greatest integer less than or equal to x.

That is, f_n is constructed by considering the first n steps of the walk and letting $f_n(j/n) = \frac{1}{\sqrt{n}} S_j$, $j = 0, 1, \ldots, n$ and linearly extrapolating for the other values of $f_n(t)$.

Recall that $\mathcal{C}[0,1]$ is a metric space with $\|f\| = \sup_{0 \le x \le 1} |f(x)|$, and $d(f,g) = \|f - g\|$.

A sequence $\{f_n\}$ converges to f wrt the metric of $\mathcal{C}[0,1]$ iff $f_n \to f$ uniformly on $[0,1]$.

Indeed, $f_n(1)$ does not converge pointwise; therefore f_n cannot converge uniformly.

A different notion of convergence is needed.

[Notice that a "scaled version of f_n is embedded in f_m, $m > n$."

Now as $n \to \infty$, these discrete approximations approach a continuous-time, continuous-space process.]

Set

$$B^n_{j/n} = f_n(j/n) = \frac{1}{\sqrt{n}} S_j$$

and

$$B^n_t = f_n(t).$$

By the CLT, the distribution of $B^n_1 = \frac{1}{\sqrt{n}} S_n$ approaches a normal distribution with mean 0 and variance 1.

Similarly, the distribution of

$$B^n_t \xrightarrow{D} \mathcal{N}(0, t).$$

The limiting process can be shown to be Brownian motion.

[Agian this is a sophisticated type of limit being taken here —convergence in distribution of random variables.]

Continuity and Differentiability

Question. How rough is the path of Brownian motion?

Consider a small increment $B(t+\Delta t) - B(t)$. By the definition of Brownian motion, this is normally distributed with mean 0 and variance Δt.

Thus,

$$\mathbb{E}(|B(t+\Delta t) - B(t)|^2) = \Delta t.$$

i.e., the typical size of an increment $|B(t+\Delta t) - B(t)|$ is about $\sqrt{\Delta t}$.

Now, as $\Delta t \to 0$, $\sqrt{\Delta t} \to 0$, which is consistent with the continuity of the paths.

However, if we try to take the derivative:

$$\frac{dB_t}{dt} = \lim_{\Delta t \to 0} \frac{B(t+\Delta t) - B(t)}{\Delta t}$$

then we see that when Δt is small, the absolute value of the numerator looks like $\sqrt{\Delta t}$ which is much larger than Δt.

Therfore, this limit does not exist.

Fact. The path of a Brownian motion B_t is nowhere differentiable.

"Brownian Motion is Really Odd"

According to Baxter and Rennie, "it is worth noting just how *odd* BM really is."

- Although BM is continuous everywhere, it is (wp1) nowhere differentiable.
- BM will eventually hit any and every real value, no matter how large or how negative! It may be a million units above the axis, but it will (wp1) be back down again to 0, by some later time.
- Once BM hits zero (or any particular value), it immediately hits it again infintely often, and then again from time to time in the future.
- No matter what scale you examine BM on, it looks just the same. Brownian motion is a fractal (it is *self-similar*).

As Wilfrid Kendall notes on the complexity of Brownian paths:

> "If you run BM in two dimensions for a positive amount of time, it will write your name."

Distributional Properties of BM

Suppose that B_t, $B_0 = 0$ is a Brownian motion.

- **spatial homogeniety**
  $B_t + x$ for any $x \in \mathbb{R}$ is a BM started at x

- **symmetry**
  $-B_t$ is a Brownian motion

- **scaling**
  $\sqrt{c}\, B_{t/c}$ for any $c > 0$ is a BM

- **time inversion**
  $Z_t = \begin{cases} 0, & t = 0, \\ t\, B_{1/t}, & t > 0. \end{cases}$ is a BM

- **time reversibility**
  for a given $t > 0$,

$$\{B_s : 0 \le s \le t\} \sim \{B_{t-s} - B_t : 0 \le s \le t\}$$

Local Path Properties of BM

- **continuity**
  Brownian paths $t \mapsto B_t$ are continuous, a.s.

- **Hölder continuity**
  $t \mapsto B_t$ is Hölder continuous of order α for every $\alpha < 1/2$. That is, $\forall\ T > 0$, $0 < \alpha < 1/2$, $\exists$ a constant $C = C(T, \alpha)$ s.t. $\forall\ t$, $s < T$,
  $$|B_t - B_s| \leq C|t - s|^{\alpha}.$$

- **nowhere differentiability**
  BM paths are a.s. nowhere Hölder continuous of order $\alpha > 1/2$. In particular, BM paths are nowhere differentiable.

- **local maxima and minima**
  For a continuous function $f : [0, \infty) \to \mathbb{R}$, a point t is a local (strict) maximum if $\exists\ \epsilon > 0$ s.t. $\forall\ s \in (t - \epsilon, t + \epsilon)$, $f(s) \leq f(t)$ $\big(f(s) < f(t)\big)$.
  Then for almost all paths, the set of local maxima for the Brownian path $B.$ is countable and dense in $[0, \infty)$, each local max is strict, and there is no interval on which the path is monotone.

- **points of increase and decrease**
  A point t is a point of increase if $\exists\ \epsilon > 0$ s.t. $\forall\ s \in (0, \epsilon)$, $f(t - s) \leq f(t) \leq f(t + s)$.
  Then for almost all paths, the Brownian path $B.$ has no points of increase or decrease.

Zero Set

Let $\mathcal{Z} = \{t : B_t = 0\}$ be the zero set of Brownian motion.

$\mathcal{Z}$ is a random set. It is almost surely unbounded and of Lebesgue measure (length) 0.

It is closed and has no isolated points i.e., it is dense in itself (such a set is called perfect).

The fractal dimension of $\mathcal{Z}$ is $1/2$.

Its complement $\mathcal{Z}^c$ is a countable union of open intervals.

From a topological perspective, $\mathcal{Z}$ looks like the Cantor set [see Rudin].

Einstein and Brownian Motion

We summarize Einstein's original argument.

Suppose there are K suspended particles in a liquid.

In a short time interval T, the x-coordinates of a single particle will increase by ϵ, where ϵ has a different value for each particle.

For the value of ϵ, a certain probability law will hold.

In the time interval T, the number dK of particles which experience a displacement between ϵ and $\epsilon + \Delta\epsilon$ can be expressed by the equation

$$dK = K\varphi(\epsilon)\, d\epsilon$$

where φ only differs from 0 for very small values of ϵ, and

$$\int_{-\infty}^{\infty} \varphi(\epsilon)\, d\epsilon = 1; \quad \varphi(\epsilon) = \varphi(-\epsilon).$$

Investigate how the coefficient of diffusion depends on φ confined to the case where the number of particles per unit volume depends on x and t only.

Set $f(x, t)$ to be the number of particles per unit volume at location x at time t. Thus, $\int_{-\infty}^{\infty} f(x, t)\, dx = K$.

By the evenness of φ, $\displaystyle\int_{-\infty}^{\infty} \epsilon\,\varphi(\epsilon)\,d\epsilon = 0.$

Define

$$\alpha^2 := \frac{1}{2T}\int_{-\infty}^{\infty} \epsilon^2\,\varphi(\epsilon)\,d\epsilon$$

We calculate the distribution of particles a short time later.

By the definition of $\varphi(\epsilon)$, we have

$$f(x,t+T) = \int_{-\infty}^{\infty} f(x+\epsilon,t)\,\varphi(\epsilon)\,d\epsilon. \qquad (*)$$

By Taylor's Theorem (ignoring higher infinitesmals), we have in one case:

$$f(x,t+T) = f(x,t) + \frac{\partial f}{\partial t}T. \qquad (1)$$

In the other case:

$$f(x+\epsilon,t) = f(x,t) + \frac{\partial f}{\partial x}\epsilon + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon^2.$$

Substituting into $(*)$ yields:

$$\begin{aligned}
f(x,t+T) &= \int_{-\infty}^{\infty} f(x+\epsilon,t)\,\varphi(\epsilon)\,d\epsilon \\
&= \int_{-\infty}^{\infty} \left[f(x,t) + \frac{\partial f}{\partial x}\epsilon + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon^2\right]\,\varphi(\epsilon)\,d\epsilon \\
&= f(x,t) + \alpha^2\frac{\partial^2 f}{\partial x^2}T \qquad (2)
\end{aligned}$$

using the properties of $\varphi(\epsilon)$ above.

By equating the two approximations, one with respect to time (1), and the other with respect to (random) displacements (2), we obtain the partial differential equation

$$\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2}$$

which is the well-known PDE for diffusion where α^2 is the coefficient of diffusion.

From this it may be concluded that

$$f(x,t) = \frac{K}{\alpha\sqrt{4\pi t}} \exp(-\frac{x^2}{4\alpha^2 t}).$$

Einstein followed the standard assumption in statistical mechanics that the "movements of the single particles are mutually independent," and that the "movements executed by a particle in consecutive time intervals are independent."

Naturally, the path of the particle is continuous.

[Notice that the formula for $f(x,t)$ is K times a $\mathcal{N}(0, 2\alpha^2 t)$ density function.]

Heat Equation and Brownian Motion

The PDE above is known as the heat conduction equation.

It arises in the following physical situation.

Consider a one dimensional rod of length L made of a single homogeneous conducting material.

PICTURE

Assume:

- sides of bar are perfectly insulated so that no heat passes through them,
- cross-sections are so small that temperature u can be considered constant on any of them

Then: variation in temperature of bar governed by heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

where α^2 is thermal diffusivity (i.e., coefficient of diffusion).

We assume an initial temperature distribution in the bar

$$u(x,0) = F(x), \quad 0 \le x \le L$$

and that the ends of the bar are held at fixed temperatures

$$u(0,t) = u(L,t) = 0, \quad t > 0.$$

Solving the heat equation is to find the function $u(x,t)$ that satisifes the heat conduction equation and given boundary conditions.

In undergraduate differential equations classes, (eg: Boyce and DiPrima), there is often a discussion of the solution of this PDE by separation of variables.

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(\frac{-n^2\pi^2\alpha^2 t}{L}) \sin\frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L F(x) \sin\frac{n\pi x}{L}\, dx.$$

Motivated by the Einstein argument, it seems natural to guess that Brownian motion should solve the heat equation.

Fact. The unique bounded solution to the heat equation is given by

$$\boxed{u(x,t) = \mathbb{E}^x \left[F(B_t) \cdot I(t,\tau) \right]}$$

where

$$I(t,\tau) = \begin{cases} 1, & t < \tau, \\ 0, & \text{otherwise,} \end{cases}$$

and τ is the (random) first time that BM hits either end of the rod.

References

[1] Martin Baxter and Andrew Rennie. *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge University Press, 1996.

[2] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems, Sixth Edition*. Wiley, 1997.

[3] Kai Lai Chung. *Green, Brown, and Probability.* World Scientific, 1995.

[4] Albert Einstein. *Investigations on the Theory of the Brownian Movement*. Dover, 1956. *(Translated from German by A. D. Cowper.)*

[5] G.R. Grimmett and D.R. Stirzaker. *Probability and Random Processes, Second Edition*. Oxford University Press, 1992.

[6] Ioannis Karatzas and Steven S. Shreve. *Brownian Motion and Stochastic Calculus, Second Edition*. Springer-Verlag, 1991.

[7] Gregory F. Lawler. *Introduction to Stochastic Processes*. Chapman & Hall, 1995.