Notes from *A Random Look at Brownian Motion*

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Normal Distribution

The formula

$$n(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

describes a "bell-curve" centred at μ with variance σ^2 (or spread σ).

A random variable N is normally distributed with mean μ and variance σ^2, written $\mathcal{N}(\mu, \sigma^2)$, if N has this density.

That is, if

$$\Pr\{N \le x\} = \int_{-\infty}^{x} n(y)\, dy = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{(y-\mu)^2}{2\sigma^2})\, dy.$$

Central Limit Theorem

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} = \frac{S_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0,1)$$

That is, the distribution of our random walk, normalized by $\sqrt{n}$, converges to the distribution of a normal random variable.

If $A \subseteq \mathbb{R}$ open interval, then

$$\lim_{n\to\infty} \Pr\{\frac{S_n}{\sqrt{n}} \in A\} = \frac{1}{\sqrt{2\pi}} \int_A \exp(-\frac{y^2}{2})\, dy.$$

Brownian Motion

A one-dimensional real-valued stochastic process $\{B_t, t \ge 0\}$ is a Brownian motion if

- $B_0 = 0$ and the function $t \mapsto B_t$ is continuous (with probability one),
- for any $t_0 < t_1 < \ldots < t_n$ the increments $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_{n-1}} - B_{t_n}$ are independent
- for any s, $t \ge 0$, the increment $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ is normally distributed.

Online Resources

Eric Weisstein's World of Mathematics

- http://scienceworld.wolfram.com/physics/BrownianMotion.html
- http://mathworld.wolfram.com/RandomWalk1-Dimensional.html
- http://mathworld.wolfram.com/WeierstrassFunction.html

Brownian motion applets illustrating various distributional properties

- http://www.stat.umn.edu/∼charlie/Stoch/brown.html
- http://www.ms.uky.edu/∼mai/java/stat/brmo.html

Some information about Robert Brown

- http://dbhs.wvusd.k12.ca.us/Chem-History/Brown-1829.html

A Brownian motion video! from the BBC

- http://www.bbc.co.uk/science/scienceshack/backcat/multimedia/vibrownianmotion.shtml

Math Talks for Undergraduates (Nam-Gyu Kang)

- http://www.math.yale.edu/∼nk36/bm.html

References

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[6] Gregory F. Lawler. *Introduction to Stochastic Processes*. Chapman & Hall, 1995.

[7] Walter Rudin. *Principles of Mathematical Analysis, Third Edition*. Wiley, 1976.