The Schramm-Loewner Evolution and Statistical Mechanics

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Colloquium
Department of Mathematics & Statistics
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March 13, 2009

Disclaimer: Everything we do will be two-dimensional: $\mathbb{C} \cong \mathbb{R}^2$

SLE-CFT

The stochastic Loewner evolution with parameter $\kappa \geq 0$ is a one-parameter family of random conformally invariant curves in the complex plane $\mathbb{C}$ invented by Oded Schramm in 1999 while considering possible scaling limits of loop-erased random walk.

Since then, it has successfully been used to study a number of lattice models from two-dimensional statistical mechanics including percolation, uniform spanning trees, self-avoiding walk, and the Ising model.

In general, there is understanding of how SLE can be used to formalize two-dimensional conformal field theory, but nevertheless there is still a lot of work to be done.

In particular, the links between SLE and two-dimensional turbulence, spin glasses, and quantum gravity are currently being investigated.

SLE-CFT

Conformal field theory (CFT) relies on the concept of a local field and its correlations in order to generate predictions about the model under consideration.

Briefly, in CFT, the central charge $\mathbf{c}$ plays a key role in delimiting the universality classes of a variety of lattice model scaling limits.

We now know that the SLE parameter κ and the central charge $\mathbf{c}$ are related through

$$\mathbf{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

Example: Self-Avoiding Walk

The self-avoiding walk was introduced in 1949 by the Nobel-prize winning chemist Paul Flory as a model of (single linear) polymer growth (in a good solvent).

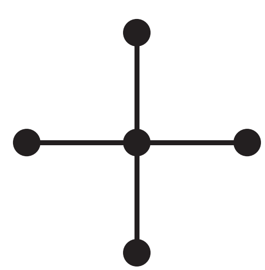
Briefly, a polymer consists of N monomers which can attach themselves to the existing chain at only certain angles. However, once a monomer occupies a site no other monomer may attach itself there. This is the so-called **excluded volume principle** which causes the polymer to repel itself.

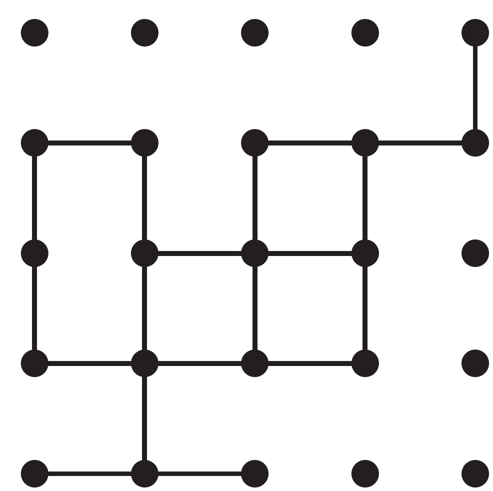
The flexibility of the polymer is modelled by the possible configurations of the self-avoiding walk, while the self-avoidance constraint models the excluded volume constraint.

Simple Random Walk

An N-step simple random walk starting at $x \in \mathbb{Z}^2$ is a path $S = [S_0, S_1, \ldots, S_N]$ with $S_0 = x$, $S_i \in \mathbb{Z}^2$, and $|S_i - S_{i-1}| = 1$.

The random walker starts at x, and at each step chooses one of its 4 nearest neighbours with equal probability and moves to that site.





Therefore, if Γ_N denotes the number of N-step simple random walk paths,

$$\Gamma_N = 4^N.$$

Furthermore, the mean-squared displacement is

$$\mathbb{E}(|S_N|^2) = N = N^{2 \cdot \frac{1}{2}}$$

where the expectation is taken with respect to the uniform measure on all N-step simple random walk paths. (Note that the proof is really easy.)

Theorem. (Donsker 1951) Simple random walk converges to Brownian motion in the scaling limit. That is, if we define the random continuous function X_N from $[0, 1]$ into $\mathbb{C}$ by setting

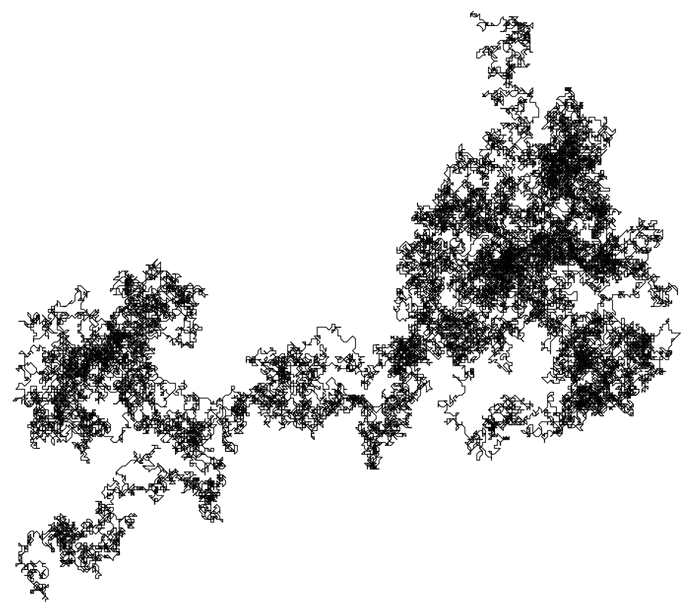
$$X_N\Big(\tfrac{j}{N}\Big) = \frac{1}{\sqrt{N}}\, S_j$$

for integers $j = 0, 1, \ldots, N$, and linearly interpolating between consecutive vertices, then the distribution of X_N converges weakly (in law or in distribution) to Brownian motion (Wiener measure).

That is, BM is the law of the scaling limit $\lim_{N \to \infty} N^{-\frac{1}{2}} S$.

Simulation of a planar Brownian motion

Brownian motion is a model of random, continuous motion. Einstein “proved” its existence along with the existence of atoms in 1905. Norbert Wiener proved its existence as a rigorous mathematical object in 1923.



Figures courtesy Lawler–Schramm–Werner 2001, 2003

An N-step self avoiding walk starting at $x \in \mathbb{Z}^2$ is a path $\mathbf{w} = [\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_N]$ with $\mathbf{w}_0 = x$, $\mathbf{w}_i \in \mathbb{Z}^2$, $|\mathbf{w}_i - \mathbf{w}_{i-1}| = 1$, and $\boxed{\mathbf{w}_i \neq \mathbf{w}_j \text{ for all } i \neq j}$.

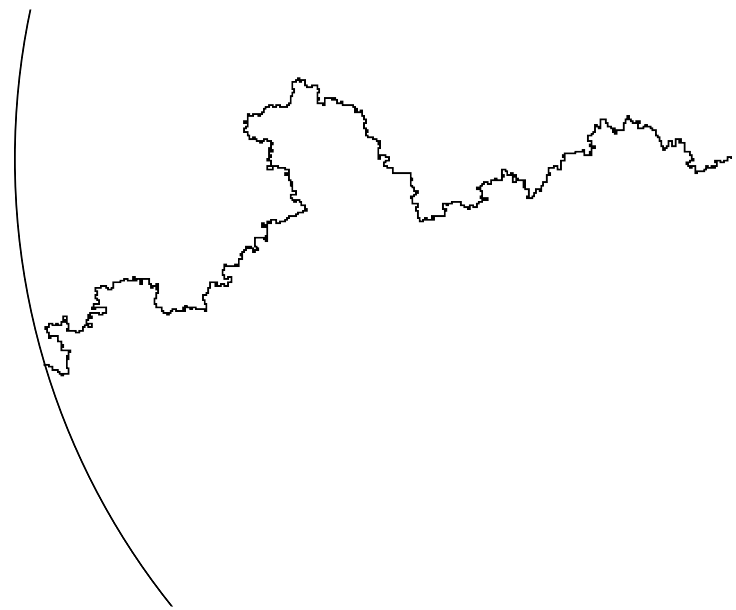


Figure courtesy Lawler–Schramm–Werner 2004

Let Ω_N denote the set of self-avoiding walks of length N starting at 0, and let $C_N := |\Omega_N|$ denote the cardinality of N.

Open problem. Determine C_N for all N.

Exact enumeration. (2005) $C_{71} = 4\ 190\ 893\ 020\ 903\ 935\ 054\ 619\ 120\ 005\ 916$

"Best" rigorous bounds. $2^N \leq C_N \leq 4(4-1)^{N-1}$

Lower bound: allowed to go right and up only

Upper bound: disallow immediate reversals

Theorem, Proof, and Definition.

$$C_{N+M} \leq C_N C_M \Rightarrow \log C_{N+M} \leq \log C_N + \log C_M$$

Therefore, $\log C_N$ is subadditive which implies $\mu := \lim_{N\to\infty} C_N^{1/N}$ exists in $(0,\infty)$. We call μ the **connective constant**.

"Best" rigorous bounds. $2 \leq \mu \leq 3$

Theorem. $2.6 \leq \mu \leq 2.7$ (numerically: $2.63816\ldots$)

Proved by many people between 1993–2005 using "rigorous numerical analysis."

We see that $C_N = \mu^N r(N)$ where $r(N)^{1/N} \to 1$ as $N \to \infty$.

Notation. Write $f(N) \sim g(N)$ if $\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1$.

Conjecture. Overwhelming evidence to suggest

$$C_N \sim \text{const}\, \mu^N N^{\gamma-1} \qquad (*)$$

with $\gamma = \frac{43}{32}$. i.e., $r(N) \sim \text{const}\, N^{\gamma-1}$. In fact, there is not even a proof that γ exists!

Note. γ is conjectured to be universal (lattice independent and more).

Theorem. (Hammersley–Welsh 1962)

$$\mu^N \le C_N \le \mu^N \exp\left\{K\sqrt{N}\right\}$$

for a positive constant K. This is a long way from $(*)$.

Assign probability $\mathbf{P}(\mathbf{w}) = \dfrac{1}{C_N}$ to each self-avoiding walk $\mathbf{w} \in \Omega_N$.
i.e., $\mathbf{P}$ is the uniform measure on all N-step self-avoiding walks

Open problem. Determine the mean-squared displacement

$$\mathbb{E}(|\mathbf{w}_N|^2) = \langle |\mathbf{w}_N|^2 \rangle$$

where the expectation is taken with respect to $\mathbf{P}$.

Open problem. Prove the "obvious" bounds

$$N \le \mathbb{E}(|\mathbf{w}_N|^2) \le \mathrm{const}\, N^{2-\epsilon}$$

for some $\epsilon > 0$.

Conjecture. $\mathbb{E}(|\mathbf{w}_N|^2) \sim \mathrm{const}\, N^{2\nu}$ with $\nu = 3/4$.

Note. ν is conjectured to be universal (lattice independent and more).

The value $\nu = 3/4$ was predicted by Flory, and is strongly supported by numerical simulations.

There are a number of other **critical exponents** (such as ν and γ) used to describe (functionals of) the self-avoiding walk.

Since critical exponents are universal, they are fundamentally more important than lattice-dependent quantities such as the connective constant μ.

Physicists are able to use nonrigorous renormalization group and conformal field theory techniques to predict critical exponents. However, none of these techniques give a prediction for the scaling limit.

That is, probabilists approached the problem by trying to determine a stochastic process which is the continuum limit of SAW and for which the exponents could be calculated.

Motivated by the case of SRW, we say that the scaling limit of SAW is the law of the path

$$\lim_{N \to \infty} N^{-\nu} \mathbf{w}.$$

SRW: "$\nu = 1/2$"

Theorem. (Lawler–Schramm–Werner 2004)

If the scaling limit exists and is conformally invariant, then it must be SLE with parameter $\kappa = \frac{8}{3}$.

The Ising Model

The Ising model is, perhaps, the simplest interacting many particle system in statistical mechanics. Although it had its origins in magnetism, it is now of importance in the context of phase transitions.

Suppose that $D \subset \mathbb{C}$ is a bounded, simply connected domain with Jordan boundary.

Consider the discrete approximation given by $D \cap \mathbb{Z}^2$.

Assign to each vertex of the square lattice a spin — either up $(+1)$ or down (-1).

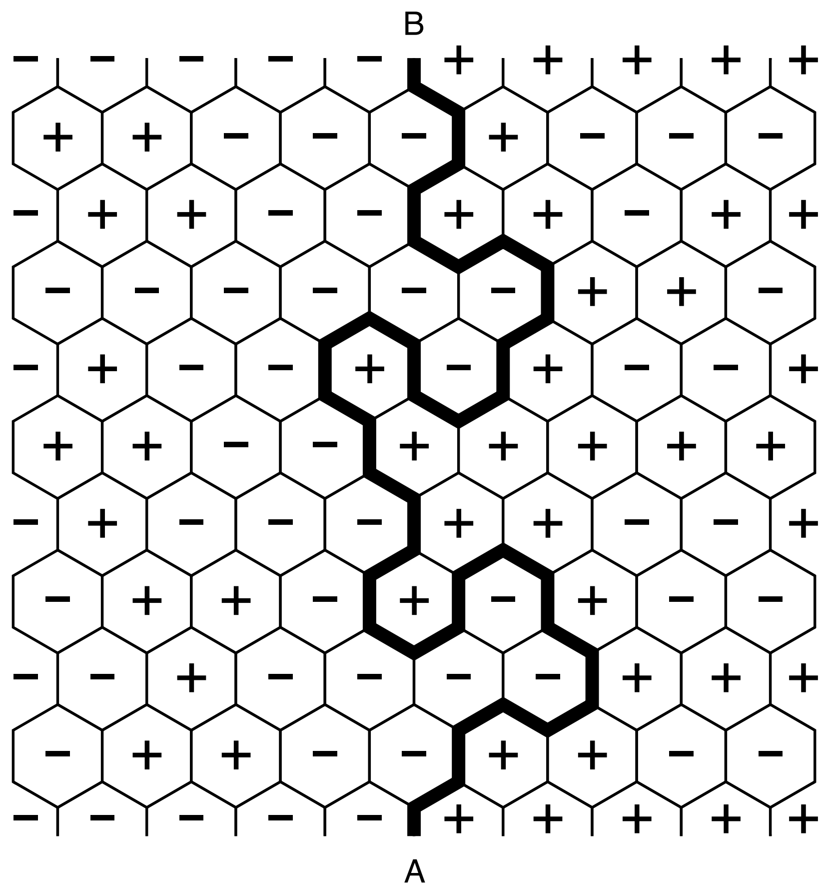
Let ω denote a configuration of spins; i.e., an element of $\Omega = \{-1, +1\}^N$ where N is the number of vertices.

Associate to the configuration the Hamiltonian

$$H(\omega) = -\sum_{i \sim j} \sigma_i \sigma_j$$

where the sum is over all nearest neighbours and $\sigma_i \in \{-1, +1\}$.

The Ising Model



The Ising Model

Define a probability measure

$$P(\{\omega\}) = \frac{\exp\{-\beta H(\omega)\}}{Z}$$

where $\beta > 0$ is a parameter and

$$Z = \sum_{\omega} \exp\{-\beta H(\omega)\}$$

is the partition function (or normalizing constant).

The parameter β is the inverse-temperature $\beta = 1/T$. It is known that there is a critical temperature T_c which separates the ferromagnetic ordered phase (below T_c) from the paramagnetic disordered phase (above T_c).

Furthermore, many physical properties (i.e., observables), such as the thermodynamic free energy, entropy, magnetization, and spin-spin correlation can be determined from the partition function.

The Ising Model

Traditionally, scaling limits in CFT are described by critical exponents.

For example, the spin-spin correlation

$$\langle \sigma_i, \sigma_j \rangle = \sum_{\omega} \sigma_i \sigma_j P(\{\omega\}) \sim \frac{\exp\{-|i-j|/\xi\}}{|i-j|^{\eta}}$$

where the correlation length ξ scales like

$$\xi \sim |T - T_c|^{-\nu}.$$

At T_c, the correlation length ξ diverges, the Ising model becomes scale invariant, and we have

$$\langle \sigma_i, \sigma_j \rangle \sim |i-j|^{-\eta}.$$

Example: The Ising Model on the Square Lattice

The point-of-view introduced by SLE is that of an interface.

Consider fixing two arcs on the boundary of the domain and holding one boundary arc all at spin up and the other all at spin down.

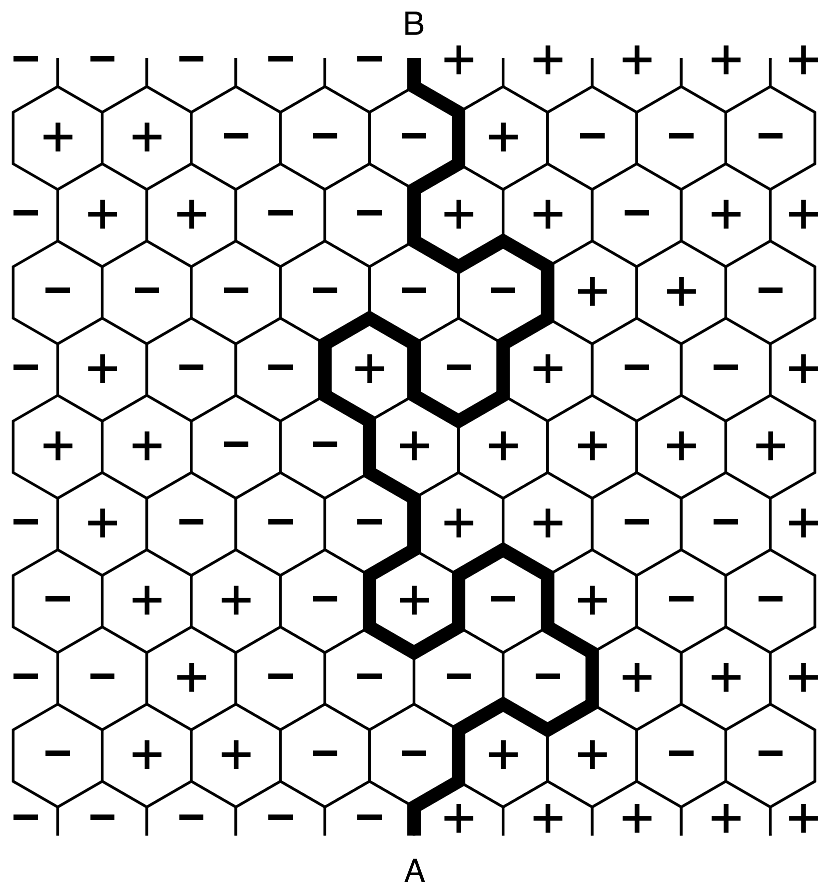
$P(\{\omega\})$ now induces a probability measure on curves (interfaces) connecting the two boundary points where the boundary conditions change.

S. Smirnov has recently showed that as the lattice spacing shrinks to 0, the interfaces converge to SLE_3.

Formally, let (D, z, w) be a simply connected Jordan domain with distinguished boundary points z and w. Let $D_n = \frac{1}{n}\mathbb{Z}^2 \cap D$ denote the $1/n$-scale square lattice approximation of D, and let z_n, w_n be the corresponding boundary points of D_n, i.e., we need $(D_n, z_n, w_n) \to (D, z, w)$ in the Carathéodory sense as $n \to \infty$.

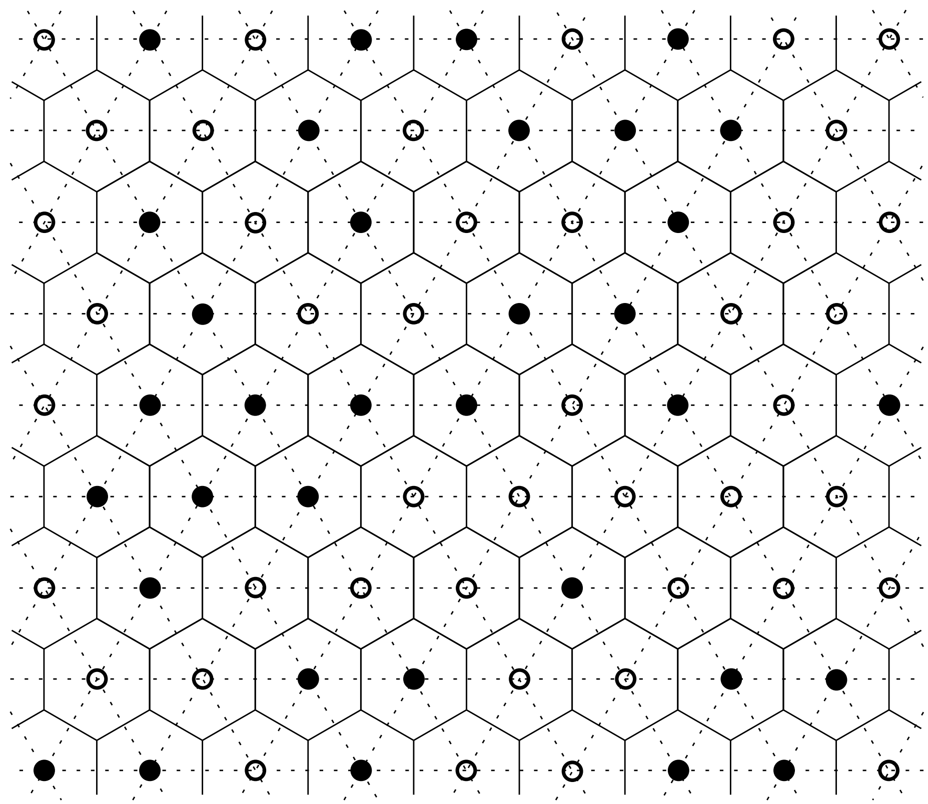
If $P_n = P_n(D_n, z_n, w_n)$ denotes the law of the discrete interface, then P_n converges weakly to $\mu_{D,z,w}$, the law of chordal SLE_3 in D from z to w.

The Ising Model



Example: Site Percolation on the Triangular Lattice

Site percolation on the triangular lattice can be identified with "face percolation" on the hexagonal lattice (which is dual to the triangular lattice).



The Discrete Percolation Exploration Path

Consider a simply connected, bounded hexagonal domain with two distinguished external vertices x and y.

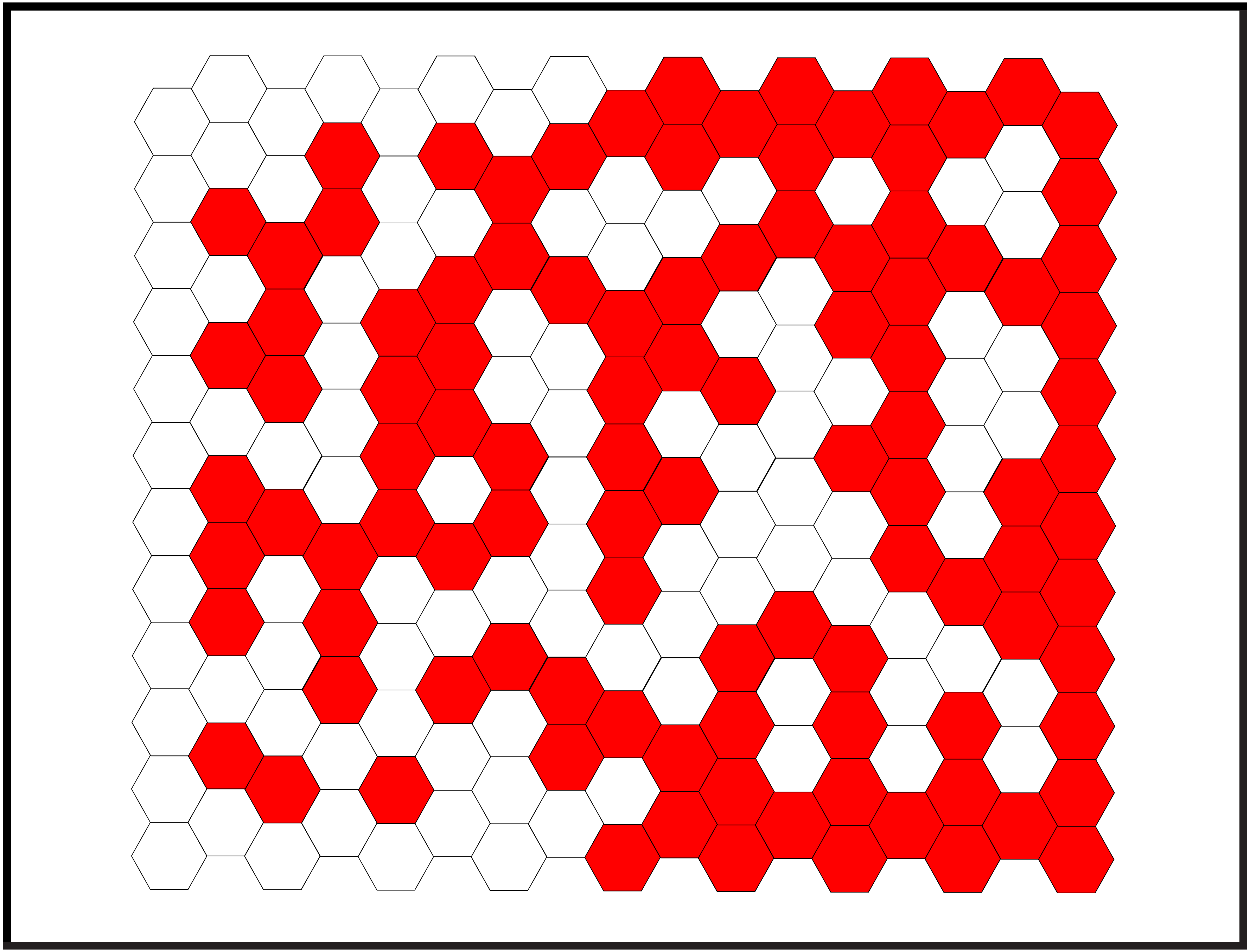
Colour all the hexagons on one half of the boundary from x to y white, and colour all the hexagons on the other half of the boundary from y to x red.

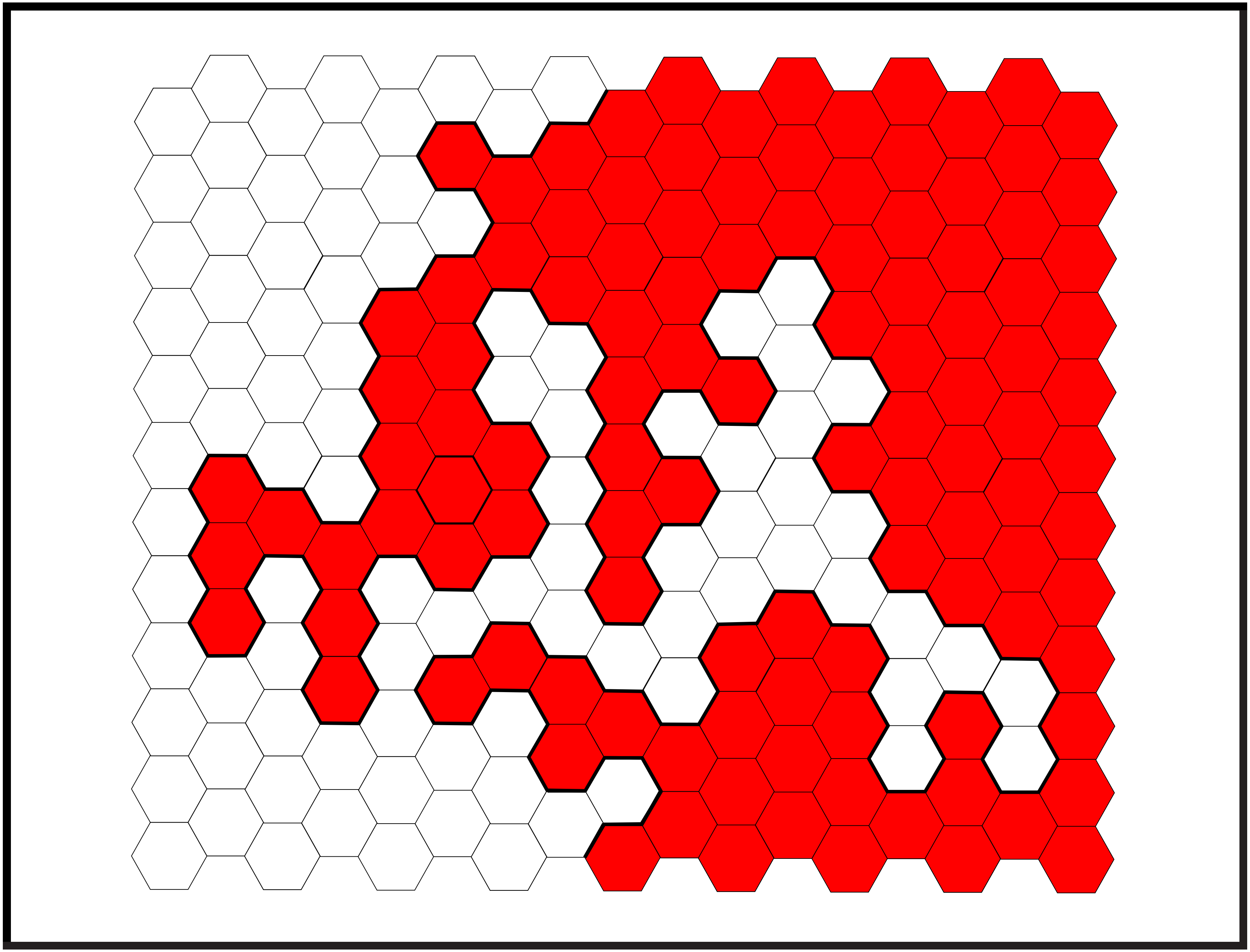
For all remaining interior hexagons colour each hexagon either red or white independently of the others each with probability 1/2 (i.e., perform critical site percolation on the triangular lattice).

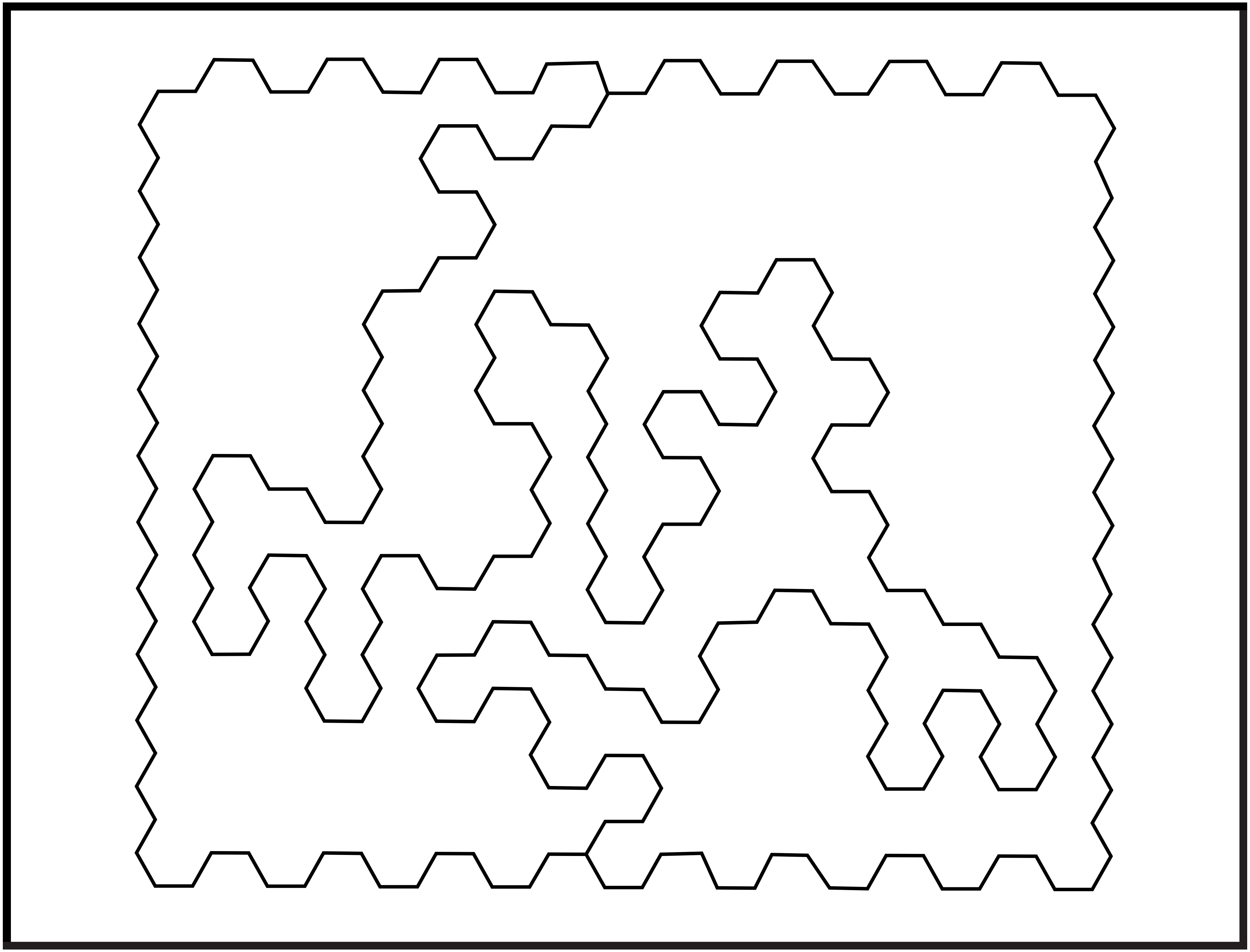
There will be an interface separating the red cluster from the white cluster.

One way is to draw the interface always keeping a red hexagon on the right and a white hexagon on the left.

Another way to visualize the interface is to swallow any islands so that the domain is partitioned into two connected sets.







The Scaling Limit of the Exploration Process

Thanks to the work of Smirnov and Werner, there is now a precise description of the scaling limit of the interface (i.e., the exploration process).

Suppose that (D, a, b) is a Jordan domain with distinguished boundary points a and b.

Let $(D^\delta, a^\delta, b^\delta)$ be a sequence of hexagonal lattice-domains with spacing δ which approximate (D, a, b).

(Technically, $(D^\delta, a^\delta, b^\delta)$ converges in the Carathéodory sense to (D, a, b) as $\delta \downarrow 0$.)

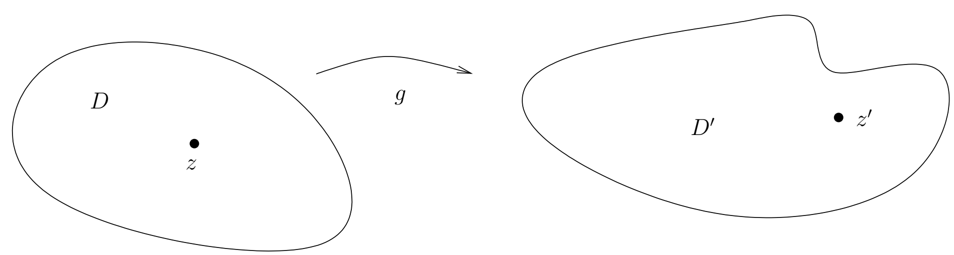
Let $\gamma^\delta = \gamma^\delta(D^\delta, a^\delta, b^\delta)$ denote the spacing δ exploration path.

As $\delta \downarrow 0$, the sequence γ^δ converges in distribution to SLE_6 in D from a to b.

Riemann Mapping Theorem

The Riemann mapping theorem (as usually presented) states that any simply connected proper subset of the complex plane can be mapped conformally onto the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem. Let D and D' be two simply connected domains each of which is a proper subset of the complex plane. Let $z \in D$ and $z' \in D'$ be two given points. Then there exists a unique analytic function g which maps D conformally onto D' and has the properties $g(z) = z'$ and $g'(z) > 0$.



It is sometimes said that *3 real degrees of freedom* uniquely specify the map.

Conformal Invariance of Brownian Motion

Brownian motion is invariant under rotations and dilations. Thus, it is no surprise that Brownian motion is conformally invariant.

Theorem. (Lévy 1948, 1967) Let $f : D \to D'$ be a conformal transformation. If B_t is a BM started at $x \in D$, stopped at τ_D, then $f(B_t)$ is a (time-changed) BM started at $f(x) \in D'$, stopped at $\tau_{D'}$.

We now have an example of a discrete model (simple random walk) and its conformally invariant scaling limit (Brownian motion).

What is SLE?

The stochastic Loewner evolution with parameter $\kappa \geq 0$ is a one-parameter family of random conformally invariant curves in the complex plane $\mathbb{C}$ invented by Schramm in 1999.

Let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a simple curve (no self intersections) with $\gamma(0) = 0$, $\gamma(0, \infty) \subseteq \mathbb{H}$, and $\gamma(t) \to \infty$ as $t \to \infty$. e.g., no "spirals"

For each $t \geq 0$ let $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ be the slit half plane and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map.

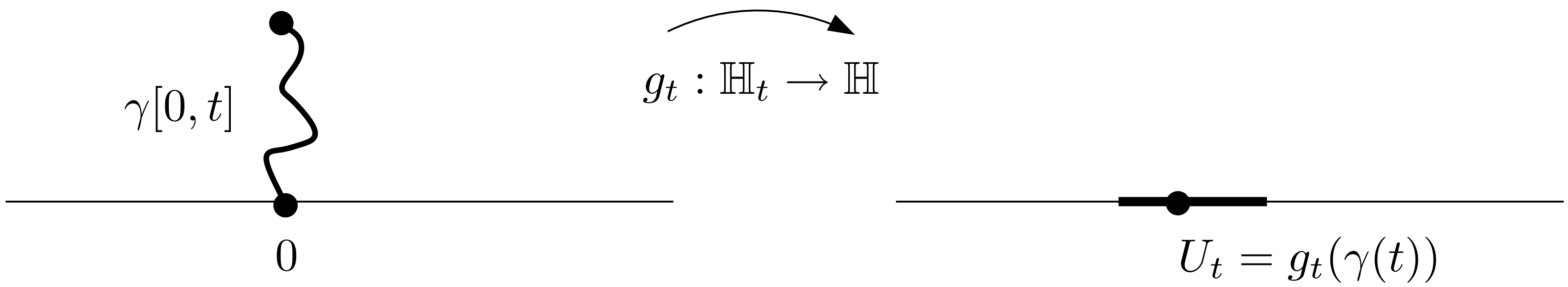
We normalize g_t and parametrize γ in such a way that as $z \to \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

Theorem. (Loewner 1923)

For fixed z, $g_t(z)$ is the solution of the IVP

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$



- The curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0,t]$, $g_t : \mathbb{H}_t \to \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0,t]) \subseteq \mathbb{R}$.

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

The natural thing to do is to start with U_t and solve the Loewner equation.

Solving the Loewner equation gives g_t which conformally map $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$.

Schramm's idea: let U_t be a Brownian motion!

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion.

Definition: SLE$_\kappa$ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

It is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. A deep theorem due to Rohde and Schramm proves this is true.

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

SLE$_\kappa$ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem. With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding $\mathbb{R}$.

- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit $\mathbb{R}$.

- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Theorem. (Beffara 2004, 2008) With probability one, the Hausdorff dimension of the SLE$_\kappa$ trace is

$$\min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$

Chordal SLE in D

Technically, we have defined **chordal SLE**. That is, SLE connecting two distinct boundary points of a simply connected domain.

Another process known as **radial SLE** connects a boundary point with an interior point.

Schramm originally defined chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. We've outlined his construction.

He then defined chordal SLE$_\kappa$ in D from z to w to be the image of SLE$_\kappa$ in $\mathbb{H}$ under a conformal transformation taking $0 \mapsto z$ and $\infty \mapsto w$.

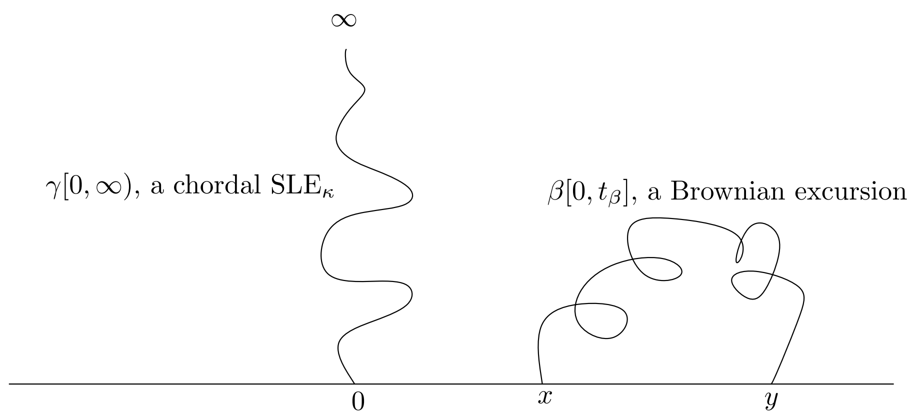
Everything is defined up to time reparametrization.

There are other constructions of chordal SLE in D. The original way could be described as the "infinitesmal approach" and uses a particular SDE. Another way is to construct a finite measure on curves via martingales and a particular Radon-Nikodym derivative.

Either way, SLE is conformally invariant.

Calculating with SLE

SLE can often be used to calculate "observables" such as crossing probabilities, critical exponents, and (non-)intersection probabilities.



Theorem. (K. 2009) Suppose that $0 < x < y < \infty$ are real numbers and let $\beta : [0, t_\beta] \to \overline{\mathbb{H}}$ be a Brownian excursion from x to y in $\mathbb{H}$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$, $0 < \kappa \le 4$, from 0 to ∞ in $\mathbb{H}$, then

$$\mathbf{P}\{\, \gamma[0,\infty) \cap \beta[0, t_\beta] = \emptyset \,\} = \frac{\Gamma(2a)\Gamma(4a+1)}{\Gamma(2a+2)\Gamma(4a-1)} \, (x/y) \, F(2, 1-2a, 2a+2; x/y)$$

where $F = {}_2F_1$ is the hypergeometric function and $a = 2/\kappa$.

Areas of Active Research

SLE describes the scaling limit of a single interface. What about multiple interfaces? This has been considered from a physical point of view by Bauer, Bernard, and Kytölä (2005). Mathematical approaches have been considered by Dubédat (2006) and by Kozdron and Lawler (2006). Rigorously constructing a measure on multiple non-crossing SLE paths for $4 < \kappa < 8$ is still an open problem.

Viewed as a mathematical object, there is interest in distributional properties of the SLE path. Beffara established the Hausdorff dimension of the curve (2004, 2008). Sheffield and Alberts (2008) determined the Hausdorff dimension of $\gamma \cap \mathbb{R}$, $4 < \kappa < 8$. It is an open problem to determine the Hausdorff dimension of the set of double-points of SLE_κ, $4 < \kappa < 8$.

There is still a lot to be done to further stregthen the links between SLE and CFT. One broad area involves rigorously proving predictions about critical exponents and other “observables” for various 2d models.