

The configurational measure on mutually avoiding SLE paths

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Charles Loewner



- 1893: born May 29 as Karl Löwner in Lany, Bohemia
- 1917: Ph.D. from University of Prague in geometric function theory under Georg Pick
- 1933: jailed during Nazi occupation of Prague, emigrated to US, changed his name to Charles Loewner, and received Assistant Professorship at Louisville University
- Brown University (1944-1946); Syracuse University (1946-1951); Stanford University (1951-1968)

Brief History

- (Loewner 1923): proved a special case of the Bieberbach conjecture ($|a_3| \leq 3$) using Loewner Equation
- (DeBranges 1985): proved entire Bieberbach conjecture
- (Schramm 1999): introduced SLE while considering scaling limits of certain stochastic processes
- (Lawler, Schramm, Werner 2000): proved Mandelbrot's conjecture that dimension of Brownian frontier is $4/3$
- (Lawler, Schramm, Werner 2006): shared SIAM's George Pólya Prize
- (Werner 2006): Fields Medal "for his contributions to development of SLE"

Riemann Mapping Theorem

The Riemann mapping theorem (as usually presented) states that any simply connected proper subset of the complex plane can be mapped conformally onto the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem: *Let D and D' be two simply connected domains each of which is a proper subset of the complex plane. Let $z \in D$ and $z' \in D'$ be two given points. Then there exists a unique analytic function g which maps D conformally onto D' and has the properties $g(z) = z'$ and $g'(z) > 0$.*

It is sometimes said that *3 real degrees of freedom* uniquely specify the map.

Slit Mappings

Let $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ be a simple curve (no self intersections) with $\gamma(0) = 0$, $\gamma(0, \infty) \subseteq \mathbb{H}$, and $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$.

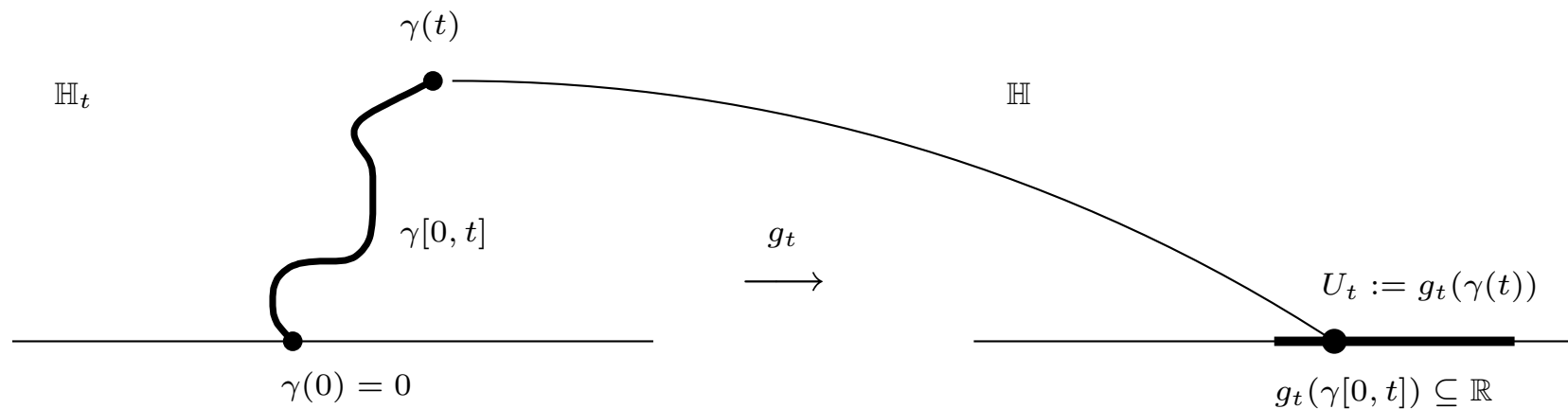
For each $t \geq 0$ let $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ be the slit half plane and let $g_t : \mathbb{H}_t \rightarrow \mathbb{H}$ be the corresponding Riemann map.

We want $g_t(\infty) = \infty$ and g_t to satisfy *hydrodynamic normalization*. (These are our 3 degrees of freedom.)

We also (re-)parametrize γ by *capacity*.

Therefore as $z \rightarrow \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$



The slit half plane \mathbb{H}_t and the corresponding Riemann map to \mathbb{H} .

- The curve $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$, $g_t : \mathbb{H}_t \rightarrow \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0, t]) \subseteq \mathbb{R}$.

The Loewner Equation

Assume that $\gamma(t)$ is parametrized by capacity.

Suppose $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ and let $g_t : \mathbb{H}_t \rightarrow \mathbb{H}$ be the corresponding maps. Let $U_t := g_t(\gamma(t))$.

Then g_t satisfies the following partial differential equation.

Theorem (Loewner 1923): *For fixed z , $g_t(z)$ is the solution of the IVP*

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

The natural thing to do is to start with U_t and solve the Loewner equation.

Suppose that the function $t \mapsto U_t$, $t \in [0, \infty)$ is continuous and real-valued.

Solving the Loewner equation gives g_t which conformally map \mathbb{H}_t to \mathbb{H} where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$. Although for many choices of U this is not possible, the following theorem gives a sufficient condition.

Theorem (Rohde-Marshall 2001): *If U is “nice” [Hölder 1/2 continuous with sufficiently small Hölder 1/2 norm], then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0, t]$.*

SLE

- **Stochastic Loewner Evolution** (aka Schramm-Loewner Evolution) introduced by Oded Schramm in 1999

The idea: let U_t be a Brownian motion!

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa}B_t$ where B_t is a standard one dimensional Brownian motion.

Definition. SLE_κ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z.$$

As Brownian motion fails the Rohde-Marshall condition, it is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. The following theorem establishes this.

Theorem (Rohde-Schramm 2001): *There exists a curve γ associated to SLE_κ .*

(The critical case $\kappa = 8$ was proved by L-S-W later in 2001.)

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

SLE_κ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0, t]$ being generated in \mathbb{H} (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem. With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding \mathbb{R} .
- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit \mathbb{R} .
- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Chordal SLE in D

Technically, we have defined *chordal SLE*. That is, SLE connecting two distinct boundary points of a simply connected domain.

Schramm originally defined chordal SLE_κ in \mathbb{H} from 0 to ∞ . (We've followed his construction.)

He then defined chordal SLE_κ in D from z to w to be the conformal image of SLE_κ in \mathbb{H} under a conformal transformation taking $0 \mapsto z$ and $\infty \mapsto w$.

Everything is defined up to time reparametrization.

There are other constructions of chordal SLE in D . The original way could be described as the “infinitesimal approach” and uses a particular SDE. Another way is via martingales and a Radon-Nikodym derivative, but only seems to work for $\kappa \leq 4$.

The Basic Setup

- $D \subset \mathbb{C}$ simply connected, ∂D Jordan
- $z_1, \dots, z_n, w_n, \dots, w_1$ distinct points ordered counterclockwise on ∂D
- write $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{w} = (w_1, \dots, w_n)$
- fix a parameter $b \in \mathbb{R}$ (boundary scaling exponent or boundary conformal weight)

Goal: To define a measure

$$Q_{D,b,n}(\mathbf{z}, \mathbf{w})$$

on mutually avoiding n -tuples $(\gamma^1, \dots, \gamma^n)$ of simple paths in D , and satisfying certain properties:

- (1) conformal covariance
- (2) boundary perturbation
- (3) cascade relation
- (4) Markov property

Note that $\gamma^i : [0, 1] \rightarrow \mathbb{C}$ with $\gamma^i(0) = z_i$, $\gamma^i(1) = w_i$, $\gamma(0, 1) \subset D$.

Conformal Covariance

If D is analytic at \mathbf{z}, \mathbf{w} , then $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ is a non-zero, finite measure supported on n -tuples $(\gamma^1, \dots, \gamma^n)$ where γ^j is a simple curve in D connecting z_j and w_j and

$$\gamma^j \cap \gamma^k = \emptyset, \quad 1 \leq j < k \leq n.$$

Moreover, if $f : D \rightarrow f(D)$ is a conformal transformation and $f(D)$ is analytic at $f(\mathbf{z}), f(\mathbf{w})$, then

$$f \circ Q_{D,b,n}(\mathbf{z}, \mathbf{w}) = |f'(\mathbf{z})|^b |f'(\mathbf{w})|^b Q_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})) \quad (1)$$

where $f(\mathbf{z}) = (f(z_1), \dots, f(z_n))$ and $f'(\mathbf{z}) = f'(z_1) \cdots f'(z_n)$.

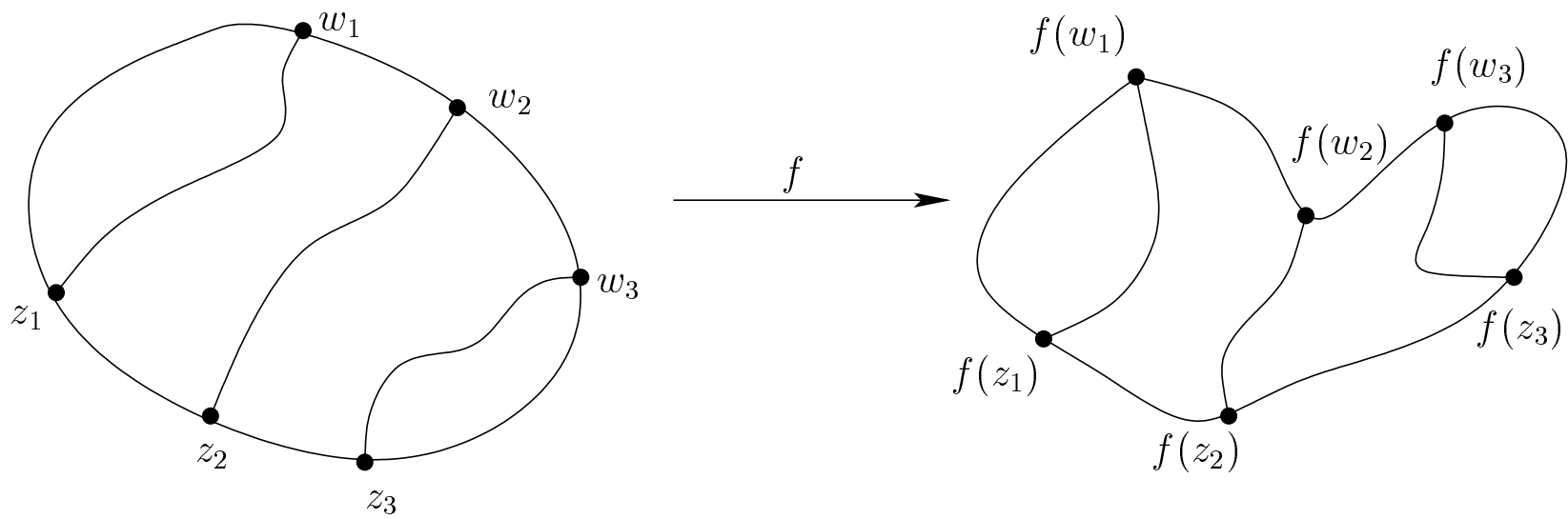


Figure 1: Conformal Covariance

Recall: $f \circ Q_{D,b,n}(\mathbf{z}, \mathbf{w}) = |f'(\mathbf{z})|^b |f'(\mathbf{w})|^b Q_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w}))$ (1)

Write

$$Q_{D,b,n}(\mathbf{z}, \mathbf{w}) = H_{D,b,n}(\mathbf{z}, \mathbf{w}) \mu_{D,b,n}^{\#}(\mathbf{z}, \mathbf{w}),$$

where $H_{D,b,n}(\mathbf{z}, \mathbf{w}) = |Q_{D,b,n}(\mathbf{z}, \mathbf{w})|$ and $\mu_{D,b,n}^{\#}(\mathbf{z}, \mathbf{w})$ is a probability measure.

The conformal covariance condition (1) then becomes the scaling rule for H ,

$$H_{D,b,n}(\mathbf{z}, \mathbf{w}) = |f'(\mathbf{z})|^b |f'(\mathbf{w})|^b H_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})),$$
 (2)

and the conformal *invariance* rule for $\mu^{\#}$,

$$f \circ \mu_{D,b,n}^{\#}(\mathbf{z}, \mathbf{w}) = \mu_{f(D),b,n}^{\#}(f(\mathbf{z}), f(\mathbf{w})).$$
 (3)

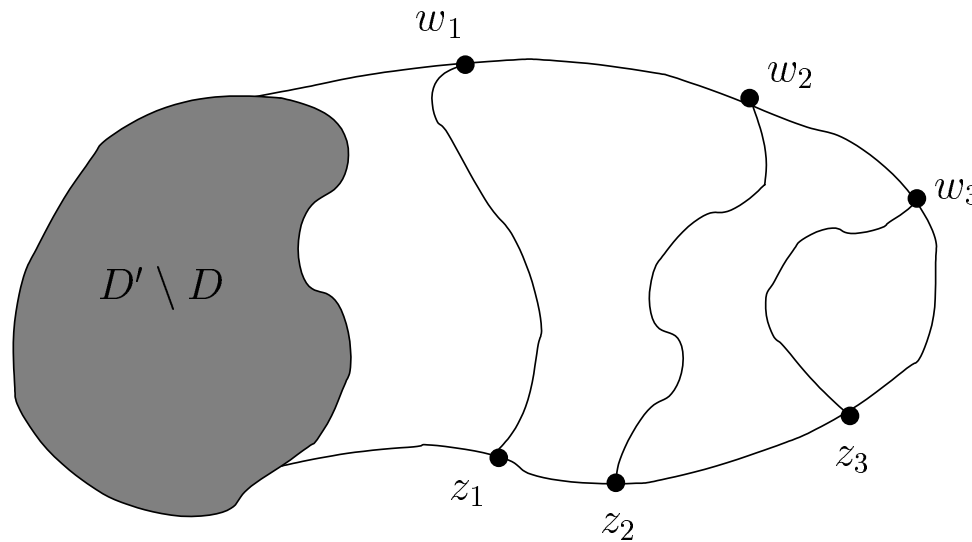
Since $\mu^{\#}$ is a conformal invariant, we can define $\mu_{D,b,n}^{\#}(\mathbf{z}, \mathbf{w})$ even if the boundaries are not smooth at \mathbf{z}, \mathbf{w} .

Boundary Perturbation

Suppose $D \subset D'$ are Jordan domains and $\partial D, \partial D'$ agree and are analytic in neighbourhoods of \mathbf{z}, \mathbf{w} . Then $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ is absolutely continuous with respect to $Q_{D',b,n}(\mathbf{z}, \mathbf{w})$. Moreover, the Radon-Nikodym derivative

$$Y_{D,D',b,n}(\mathbf{z}, \mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z}, \mathbf{w})}{dQ_{D',b,n}(\mathbf{z}, \mathbf{w})}$$

is a conformal invariant.



Recall: $D \subset D'$ and

$$Y_{D,D',b,n}(\mathbf{z}, \mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z}, \mathbf{w})}{dQ_{D',b,n}(\mathbf{z}, \mathbf{w})}$$

Saying that $Y_{D,D',b,n}(\mathbf{z}, \mathbf{w})$ is a conformal invariant means that if $f : D' \rightarrow f(D')$ is a conformal map that extends analytically in neighbourhoods of \mathbf{z}, \mathbf{w} , then

$$Y_{f(D),f(D'),b,n}(f(\mathbf{z}), f(\mathbf{w}))(f \circ \bar{\gamma}) = Y_{D,D',b,n}(\mathbf{z}, \mathbf{w})(\bar{\gamma}), \quad (4)$$

where $\bar{\gamma} = (\gamma^1, \dots, \gamma^n)$ and $f \circ \bar{\gamma} = (f \circ \gamma^1, \dots, f \circ \gamma^n)$.

As with $\mu_{D,b,n}^\#(\mathbf{z}, \mathbf{w})$, the last condition (4) implies that $Y_{D,D',b,n}(\mathbf{z}, \mathbf{w})$ is well-defined even if the boundaries are not smooth at \mathbf{z}, \mathbf{w} .

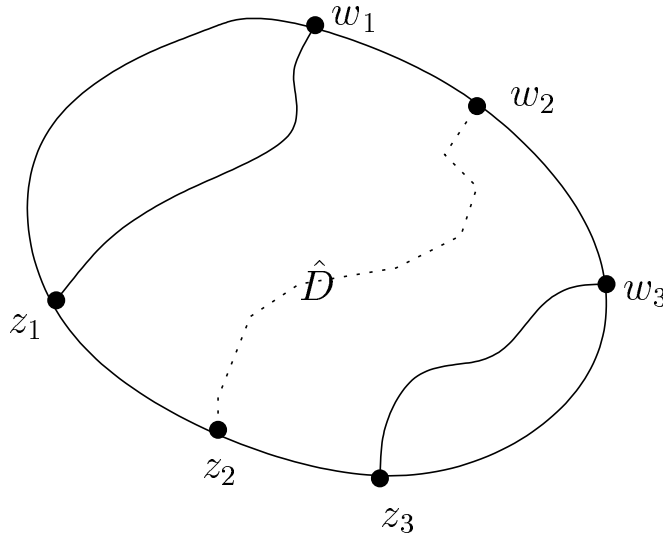
Cascade Relation

Let

$$\hat{\mathbf{z}} = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n), \quad \hat{\mathbf{w}} = (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n),$$

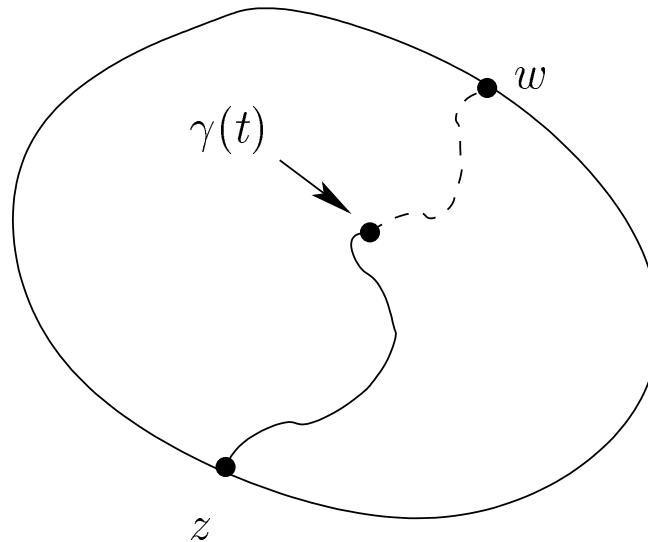
$$\hat{\gamma} = (\gamma^1, \dots, \gamma^{j-1}, \gamma^{j+1}, \dots, \gamma^n).$$

The marginal distribution on $\hat{\gamma}$ induced by $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ is absolutely continuous with respect to $Q_{D,b,n-1}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$ with Radon-Nikodym derivative $H_{\hat{D},b,1}(z_j, w_j)$. Here \hat{D} is the subdomain of $D \setminus \hat{\gamma}$ whose boundary includes z_j, w_j .



Markov Property

In the measure $\mu_{D,b,1}^\#(z,w)$, the conditional distribution on γ given an initial segment $\gamma[0,t]$ is $\mu_{D \setminus \gamma[0,t],b,1}^\#(\gamma(t),w)$.



Note: We have stated this condition in a way that does not use two dimensions and conformal invariance.

Recall: Schramm's Result

The *conformal Markov property* is the combination of the Markov property and (3). Schramm showed that there is a one-parameter family of measures, which he parametrized by κ , satisfying the conformal Markov property. While these measures are well-defined for $\kappa > 0$, they are supported on simple curves only for $0 < \kappa \leq 4$.

Existence of the Configurational Measure

Theorem (Kozdron-Lawler): For any $b \geq \frac{1}{4}$, there exists a family of measures $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n -tuples of mutually avoiding simple curves satisfying

- conformal covariance
- boundary perturbation
- cascade relation
- Markov property

Moreover, the simple curve γ^i is a chordal SLE_κ from z_i to w_i in D where

$$\kappa = \frac{6}{2b + 1}.$$

Note: $b \geq \frac{1}{4} \iff 0 < \kappa \leq 4$

The Partition Function for Two Paths

By conformal invariance, it suffices to work in $D = \mathbb{H}$.

If $0 < x_1 < \cdots < x_n < y_n < \cdots < y_1 < \infty$, let

$$H_{\mathbb{H},b,n}^*(\mathbf{x}, \mathbf{y}) = \lim_{w \rightarrow \infty} w^{2b} H_{\mathbb{H},b,n+1}((0, \mathbf{x}), (w, \mathbf{y})).$$

Proposition: If $b \geq 1/4$ and $n + 1 = 2$, then

$$H_{\mathbb{H},b,1}^*(x, y) = (y - x)^{-2b} \frac{\Gamma(2a) \Gamma(6a - 1)}{\Gamma(4a) \Gamma(4a - 1)} (x/y)^a F(2a, 1 - 2a, 4a; x/y)$$

where F denotes the hypergeometric function and $a = \frac{2}{\kappa} = \frac{2b + 1}{3}$.

Note: This result first appeared in J. Dubédat, and was derived non-rigorously by M. Bauer, D. Bernard, and K. Kytölä. Our configurational approach provides another rigorous derivation.

The Scaling Limit of Fomin's Identity

Theorem (Kozdron-Lawler): *If $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ is an SLE_2 in the upper half plane \mathbb{H} from 0 to ∞ , and $\beta : [0, 1] \rightarrow \overline{\mathbb{H}}$ is a Brownian excursion from x to y in \mathbb{H} where $0 < x < y < \infty$, then*

$$\mathbf{P} \{ \gamma[0, \infty) \cap \beta[0, 1] = \emptyset \} = 1 - \frac{H(f(0), f(y)) H(f(x), f(\infty))}{H(f(0), f(\infty)) H(f(x), f(y))}$$

where $f : \mathbb{H} \rightarrow \mathbb{D}$ is a conformal transformation of the upper half plane \mathbb{H} onto the unit disk \mathbb{D} , and $H(z, w)$ is the excursion Poisson kernel in \mathbb{D} given by

$$H(z, w) := H_{\partial\mathbb{D}}(z, w) := \frac{1}{\pi} \frac{1}{|w - z|^2} = \frac{1}{2\pi} \frac{1}{1 - \cos(\arg w - \arg z)}.$$

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