A Random Look at the Schramm-Loewner Evolution

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Abstract

The stochastic Loewner evolution (SLE) is a one-parameter family of random geometric processes in the complex plane introduced by Oded Schramm in 1999 which is believed to describe the scaling limit of a variety of statistical mechanics models. Recently a number of rigorous results about such scaling limits have been established; in fact, Wendelin Werner was awarded the Fields Medal in 2006 for "his contributions to the development of stochastic Loewner evolution, the geometry of two-dimensional Brownian motion, and conformal field theory." In this talk I will introduce SLE, describe some of its basic properties including the relationship between SLE, Brownian motion, and CFT, and illustrate its connection to a variety of models including the Ising model, self-avoiding walk, loop-erased random walk, and percolation. This talk will be "colloquium style" and is intended for a general mathematics audience.

Disclaimer: Everything we do will be two-dimensional: $\mathbb{C} \cong \mathbb{R}^2$

What is SLE?

The stochastic Loewner evolution with parameter $\kappa \geq 0$ is a one-parameter family of random conformally invariant curves in the complex plane $\mathbb{C}$ invented by Schramm in 1999.

Motivation from Statistical Mechanics

A number of simple models introduced in statistical mechanics have proven to be notoriously difficult to analyze in a rigorous mathematical way. In this talk we will discuss one of them in detail, namely the **self-avoiding walk**. In doing so, however, we will need to discuss **simple random walk** and **loop-erased random walk**. We may also touch on the **Ising model** and **percolation**.

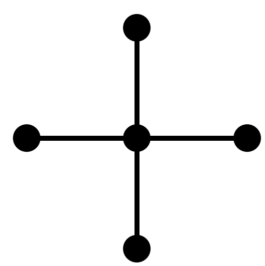
Brief History

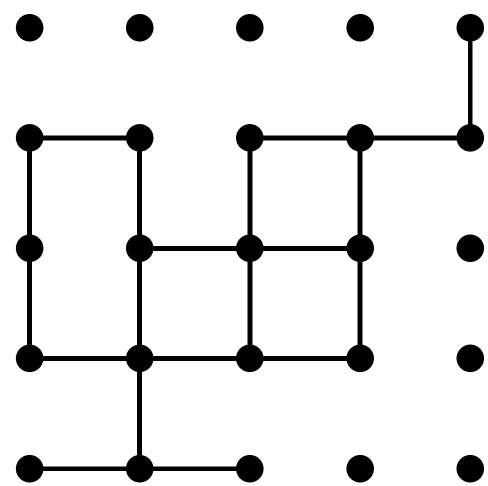
- (Loewner 1923): proved a special case of the Bieberbach conjecture ($|a_3| \leq 3$) using Loewner Equation
- (DeBranges 1985): proved entire Bieberbach conjecture
- (Schramm 1999): introduced SLE while considering scaling limits of loop-erased random walk
- (Lawler–Schramm–Werner 2000): proved Mandelbrot's conjecture that dimension of Brownian frontier is 4/3
- (Lawler–Schramm–Werner 2006): shared SIAM's George Pólya Prize
- (Werner 2006): Fields Medal "for his contributions to development of SLE"

Simple Random Walk

An N-step simple random walk starting at $x \in \mathbb{Z}^2$ is a path $S = [S_0, S_1, \ldots, S_N]$ with $S_0 = x$, $S_i \in \mathbb{Z}^2$, and $|S_i - S_{i-1}| = 1$.

The random walker starts at x, and at each step chooses one of its 4 nearest neighbours with equal probability and moves to that site.





Therefore, if Γ_N denotes the number of N-step simple random walk paths,

$$\Gamma_N = 4^N.$$

Furthermore, the mean-squared displacement is

$$\mathbf{E}(|S_N|^2) = N = N^{2 \cdot \frac{1}{2}}$$

where the expectation is taken with respect to the uniform measure on all N-step simple random walk paths. (Note that the proof is really easy.)

Theorem: (Donsker 1951) Simple random walk converges to Brownian motion in the scaling limit. That is, if we define the random continuous function X_N from $[0, 1]$ into $\mathbb{C}$ by setting

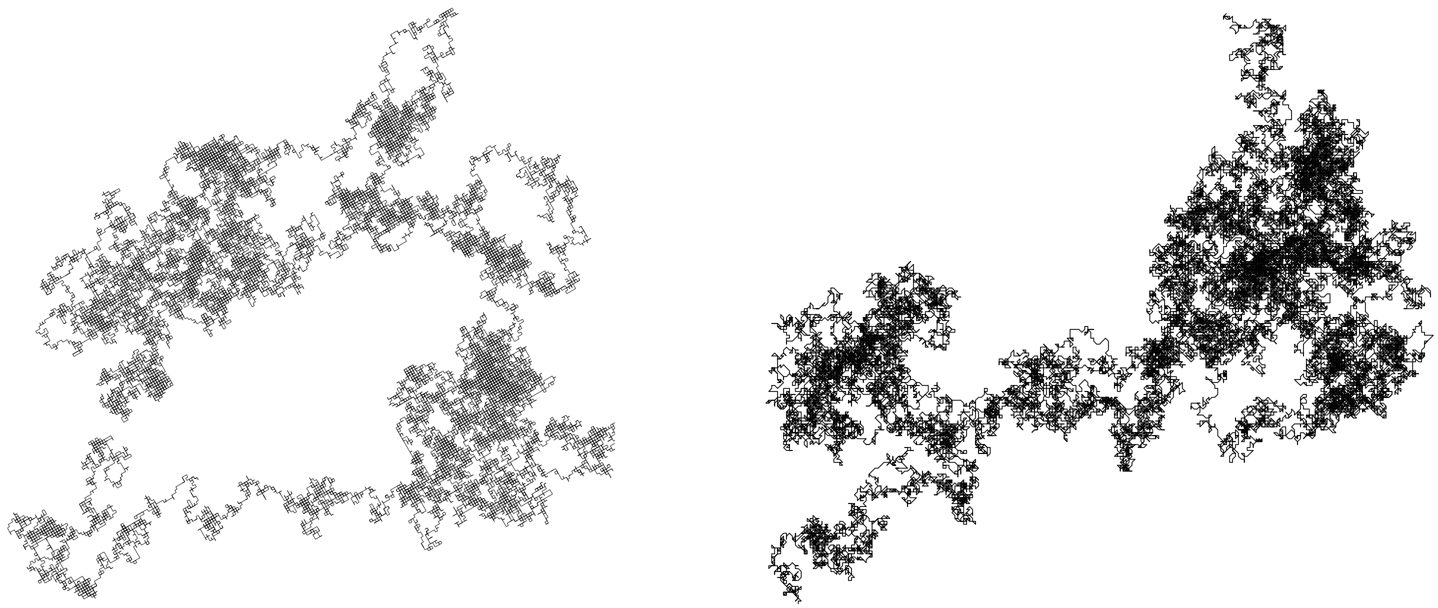
$$X_N\left(\tfrac{j}{N}\right) = \frac{1}{\sqrt{N}}\, S_j$$

for integers $j = 0, 1, \ldots, N$, and linearly interpolating between consecutive vertices, then the distribution of X_N converges weakly (in law or in distribution) to Brownian motion (Wiener measure).

That is, BM is the law of the scaling limit $\lim\limits_{N \to \infty} N^{-\frac{1}{2}} S$.

Simulation of a planar Brownian motion

Brownian motion is a model of random, continuous motion. Einstein “proved” its existence along with the existence of atoms in 1905. Norbert Wiener proved its existence as a rigorous mathematical object in 1923.



Figures courtesy Lawler–Schramm–Werner 2001, 2003

Self-Avoiding Walk

The self-avoiding walk was introduced in 1949 by the Nobel-prize winning chemist Paul Flory as a model of (single linear) polymer growth (in a good solvent).

Briefly, a polymer consists of N monomers which can attach themselves to the existing chain at only certain angles. However, once a monomer occupies a site no other monomer may attach itself there. This is the so-called **excluded volume principle** which causes the polymer to repel itself.

The flexibility of the polymer is modelled by the possible configurations of the self-avoiding walk, while the self-avoidance constraint models the excluded volume constraint.

An N-step self avoiding walk starting at $x \in \mathbb{Z}^2$ is a path $\omega = [\omega_0, \omega_1, \ldots, \omega_N]$ with $\omega_0 = x$, $\omega_i \in \mathbb{Z}^2$, $|\omega_i - \omega_{i-1}| = 1$, and $\boxed{\omega_i \neq \omega_j \text{ for all } i \neq j}$.

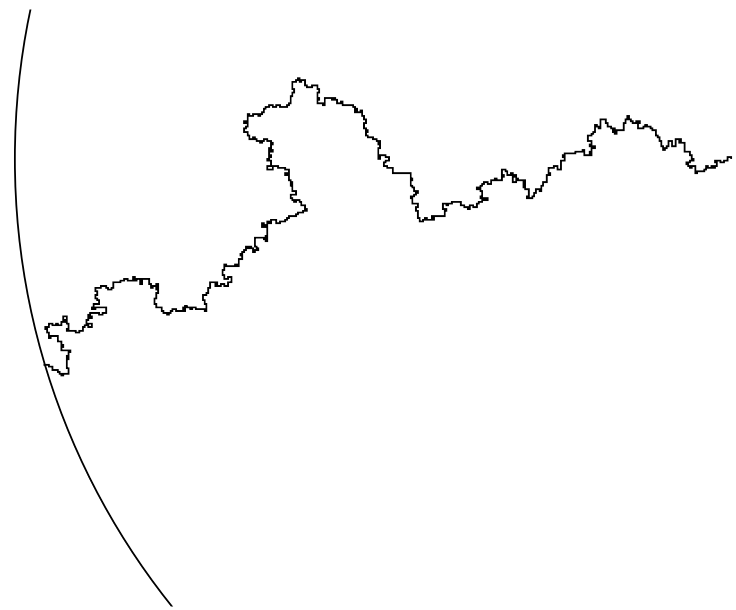


Figure courtesy Lawler–Schramm–Werner 2004

Let Ω_N denote the set of self-avoiding walks of length N starting at 0, and let $C_N := |\Omega_N|$ denote the cardinality of N.

Open problem: Determine C_N for all N.

Exact enumeration: (2005) $C_{71} = 4\ 190\ 893\ 020\ 903\ 935\ 054\ 619\ 120\ 005\ 916$

“Best” rigorous bounds: $2^N \leq C_N \leq 4(4-1)^{N-1}$

Lower bound: allowed to go right and up only

Upper bound: disallow immediate reversals

Theorem, Proof, and Definition:

$$C_{N+M} \leq C_N C_M \Rightarrow \log C_{N+M} \leq \log C_N + \log C_M$$

Therefore, $\log C_N$ is subadditive which implies $\mu := \lim_{N\to\infty} C_N^{1/N}$ exists in $(0,\infty)$. We call μ the **connective constant**.

“Best” rigorous bounds: $2 \leq \mu \leq 3$

Theorem: $2.6 \leq \mu \leq 2.7$ (numerically: $2.63816\ldots$)

Proved by many people between 1993-2005 using “rigorous numerical analysis.”

We see that $C_N = \mu^N r(N)$ where $r(N)^{1/N} \to 1$ as $N \to \infty$.

Notation: Write $f(N) \sim g(N)$ if $\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1$.

Conjecture: Overwhelming evidence to suggest

$$C_N \sim \text{const}\, \mu^N N^{\gamma - 1} \qquad (*)$$

with $\gamma = \frac{43}{32}$. i.e., $r(N) \sim \text{const}\, N^{\gamma-1}$. In fact, there is not even a proof that γ exists!

Note: γ is conjectured to be universal (lattice independent and more).

Theorem: (Hammersley–Welsh 1962)

$$\mu^N \le C_N \le \mu^N \exp\left\{K\sqrt{N}\right\}$$

for a positive constant K. This is a long way from $(*)$.

Assign probability $\mathbf{P}(\omega) = \dfrac{1}{C_N}$ to each self-avoiding walk $\omega \in \Omega_N$.
i.e., $\mathbf{P}$ is the uniform measure on all N-step self-avoiding walks

Open problem: Determine the mean-squared displacement

$$\mathbf{E}(|\omega_N|^2) = \langle |\omega_N|^2 \rangle$$

where the expectation is taken with respect to $\mathbf{P}$.

Open problem: Prove the "obvious" bounds

$$N \le \mathbf{E}(|\omega_N|^2) \le \text{const}\, N^{2-\varepsilon}$$

for some $\varepsilon > 0$.

Conjecture: $\mathbf{E}(|\omega_N|^2) \sim \text{const}\, N^{2\nu}$ with $\nu = 3/4$.

Note: ν is conjectured to be universal (lattice independent and more).

The value $\nu = 3/4$ was predicted by Flory, and is strongly supported by numerical simulations.

There are a number of other **critical exponents** (such as ν and γ) used to describe (functionals of) the self-avoiding walk.

Since critical exponents are universal, they are fundamentally more important than lattice-dependent quantities such as the connective constant μ.

Physicists are able to use nonrigorous renormalization group and conformal field theory techniques to predict critical exponents. However, none of these techniques give a prediction for the scaling limit.

That is, probabilists approached the problem by trying to determine a stochastic process which is the continuum limit of SAW and for which the exponents could be calculated.

Motivated by the case of SRW, we say that the scaling limit of SAW is the law of the path

$$\lim_{N\to\infty} N^{-\nu}\omega.$$

SRW: "$\nu = 1/2$"

Theorem: (Lawler–Schramm–Werner 2004)

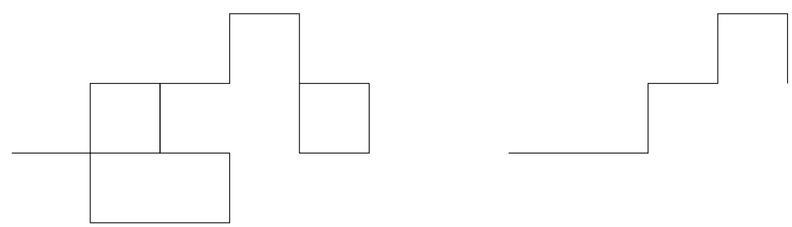
If the scaling limit exists and is conformally invariant, then it must be SLE with parameter $\kappa = \frac{8}{3}$.

Conjecture: The scaling limit exists and is conformally invariant

Loop-erased Random Walk

The loop-erased random walk was introduced by Lawler in 1980 in an attempt to analyze the SAW. However, it turned out be in a different universality class than SAW.

Chronologically erase loops from a simple random walk path. That is, if $S = [S_0, S_1, \ldots, S_N]$ is an N-step simple random walk path, erase loops chronologically to produce a self-avoiding path $\Lambda = [\Lambda_0, \Lambda_1, \ldots, \Lambda_M]$ with $M \leq N$.



The uniform measure on LERW of length M is not the same as the uniform measure on SAW of length M. (Not an obvious fact to prove.)

Theorem: (Lawler–Schramm–Werner 2004) The scaling limit of LERW is (radial) SLE with parameter $\kappa = 2$.

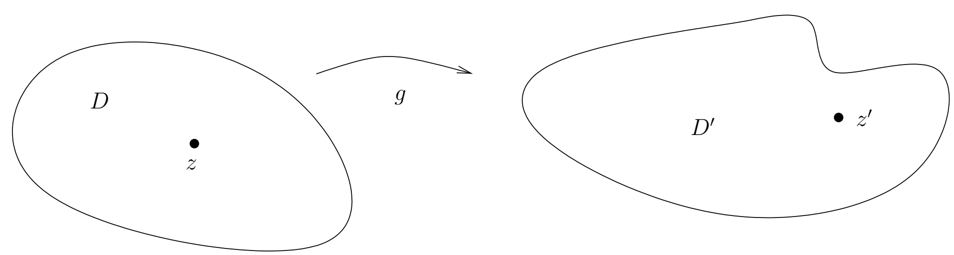
Theorem: (Zhan 2007, Johansson 200X) The scaling limit of LERW is (chordal) SLE with parameter $\kappa = 2$.

Corollary: Exact values for a number of critical exponents for LERW are no longer predictions/conjectures.

Riemann Mapping Theorem

The Riemann mapping theorem (as usually presented) states that any simply connected proper subset of the complex plane can be mapped conformally onto the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

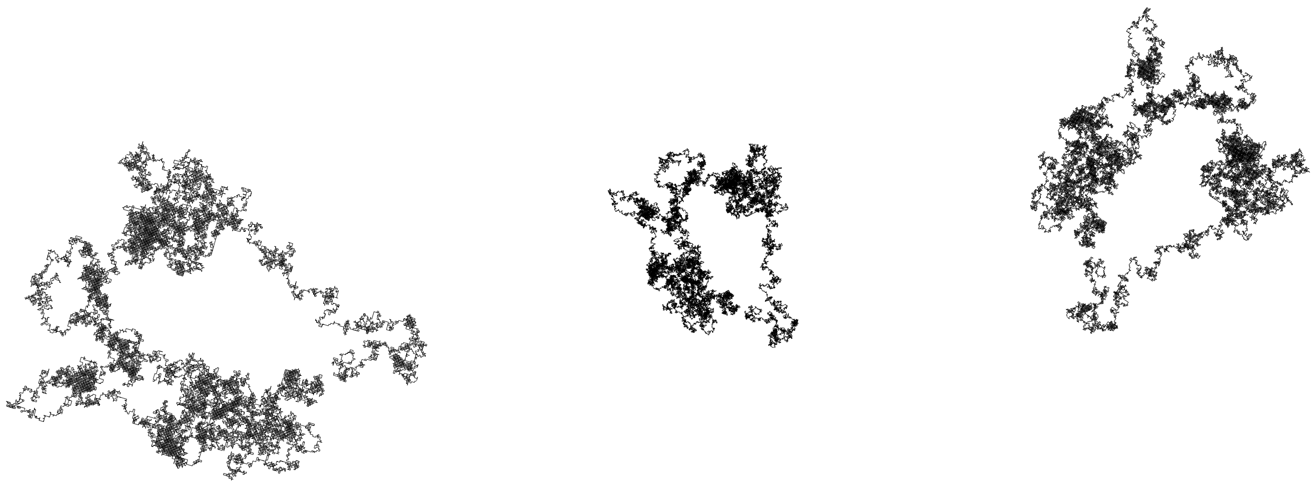
Theorem: *Let D and D' be two simply connected domains each of which is a proper subset of the complex plane. Let $z \in D$ and $z' \in D'$ be two given points. Then there exists a unique analytic function g which maps D conformally onto D' and has the properties $g(z) = z'$ and $g'(z) > 0$.*



It is sometimes said that *3 real degrees of freedom* uniquely specify the map.

Conformal Invariance of Brownian Motion

Brownian motion is invariant under rotations and dilations. Thus, it is no surprise that Brownian motion is conformally invariant.



Theorem: (Lévy 1948, 1967) Let $f : D \to D'$ be a conformal transformation. If B_t is a BM started at $x \in D$, stopped at τ_D, then $f(B_t)$ is a (time-changed) BM started at $f(x) \in D'$, stopped at $\tau_{D'}$.

We now have an example of a discrete model (simple random walk) and its conformally invariant scaling limit (Brownian motion).

Slit Mappings

Let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a simple curve (no self intersections) with $\gamma(0) = 0$, $\gamma(0, \infty) \subseteq \mathbb{H}$, and $\gamma(t) \to \infty$ as $t \to \infty$. e.g., no "spirals"

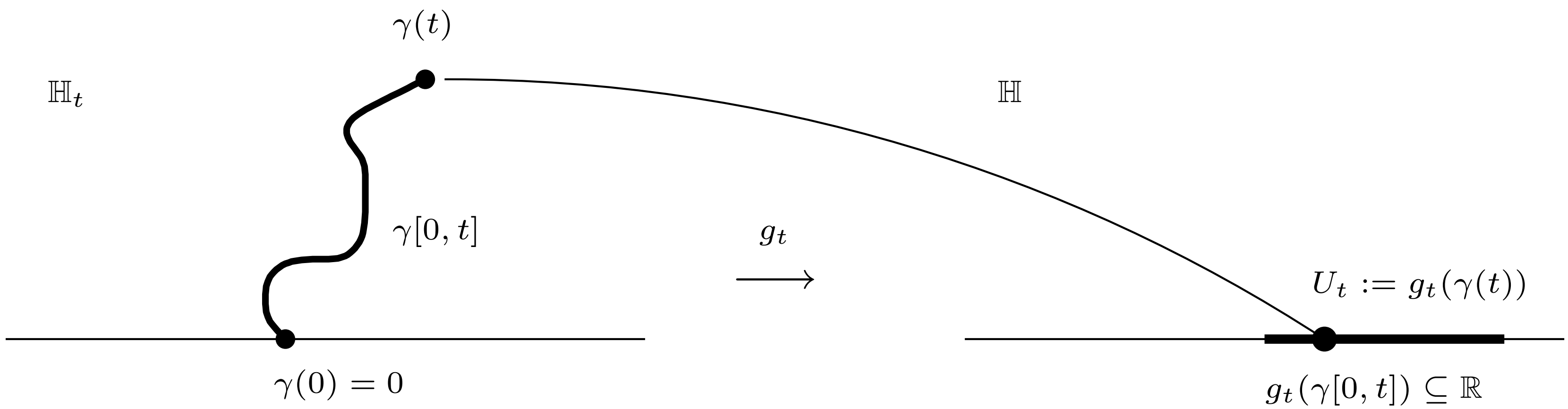
For each $t \geq 0$ let $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ be the slit half plane and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map.

We want $g_t(\infty) = \infty$ and g_t to satisfy *hydrodynamic normalization*. (These are our 3 degrees of freedom.)

We also (re-)parametrize γ by *capacity*.

Therefore as $z \to \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$



The slit half plane $\mathbb{H}_t$ and the corresponding Riemann map to $\mathbb{H}$.

- The curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0,t]$, $g_t : \mathbb{H}_t \to \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0,t]) \subseteq \mathbb{R}$.

The Loewner Equation

Assume that $\gamma(t)$ is parametrized by capacity.

Suppose $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding maps. Let $U_t := g_t(\gamma(t))$.

Then g_t satisfies the following partial differential equation.

Theorem: (Loewner 1923)

For fixed z, $g_t(z)$ is the solution of the IVP

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

The natural thing to do is to start with U_t and solve the Loewner equation.

Suppose that the function $t \mapsto U_t$, $t \in [0, \infty)$ is continuous and real-valued.

Solving the Loewner equation gives g_t which conformally map $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$. Although for many choices of U this is not possible, the following theorem gives a sufficient condition.

Theorem: (Rohde–Marshall 2001) If U is "nice" [Hölder 1/2 continuous with sufficiently small Hölder 1/2 norm], then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0, t]$.

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

Schramm's idea: let U_t be a Brownian motion!

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion.

Definition: SLE$_\kappa$ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

As Brownian motion fails the Rohde–Marshall condition, it is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. The following theorem establishes this.

Theorem: (Rohde–Schramm 2001)

There exists a curve γ associated to SLE$_\kappa$.

(The critical case $\kappa = 8$ was proved by L-S-W later in 2001.)

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

SLE$_\kappa$ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem: With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding $\mathbb{R}$.

- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit $\mathbb{R}$.

- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Theorem: With probability one, the Hausdorff dimension of the SLE$_\kappa$ trace is

$$\min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$

Note: If $\kappa = \frac{8}{3}$, then γ is a simple curve with $\dim \gamma = 1 + \frac{\kappa}{8} = \frac{4}{3}$.

Chordal SLE in D

Technically, we have defined **chordal SLE**. That is, SLE connecting two distinct boundary points of a simply connected domain.

Another process known as **radial SLE** connects a boundary point with an interior point.

Schramm originally defined chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. We've outlined his construction.

He then defined chordal SLE$_\kappa$ in D from z to w to be the image of SLE$_\kappa$ in $\mathbb{H}$ under a conformal transformation taking $0 \mapsto z$ and $\infty \mapsto w$.

Everything is defined up to time reparametrization.

There are other constructions of chordal SLE in D. The original way could be described as the "infinitesmal approach" and uses a particular SDE. Another way is via martingales and a Radon-Nikodym derivative, but only seems to work for $\kappa \leq 4$.

In order to construct "multiple SLEs in a simply connected domain" much more care is needed. This has been done by Dubedat (2004) modifying the "infinitesmal approach" and by Kozdron–Lawler (2007) using a "configurational approach."

Current State of Scaling Limit Results

Let $D \subsetneq \mathbb{C}$ be a simply connected Jordan domain. For ease, assume that ∂D is "sufficiently smooth." (e.g., a rectangle works well)

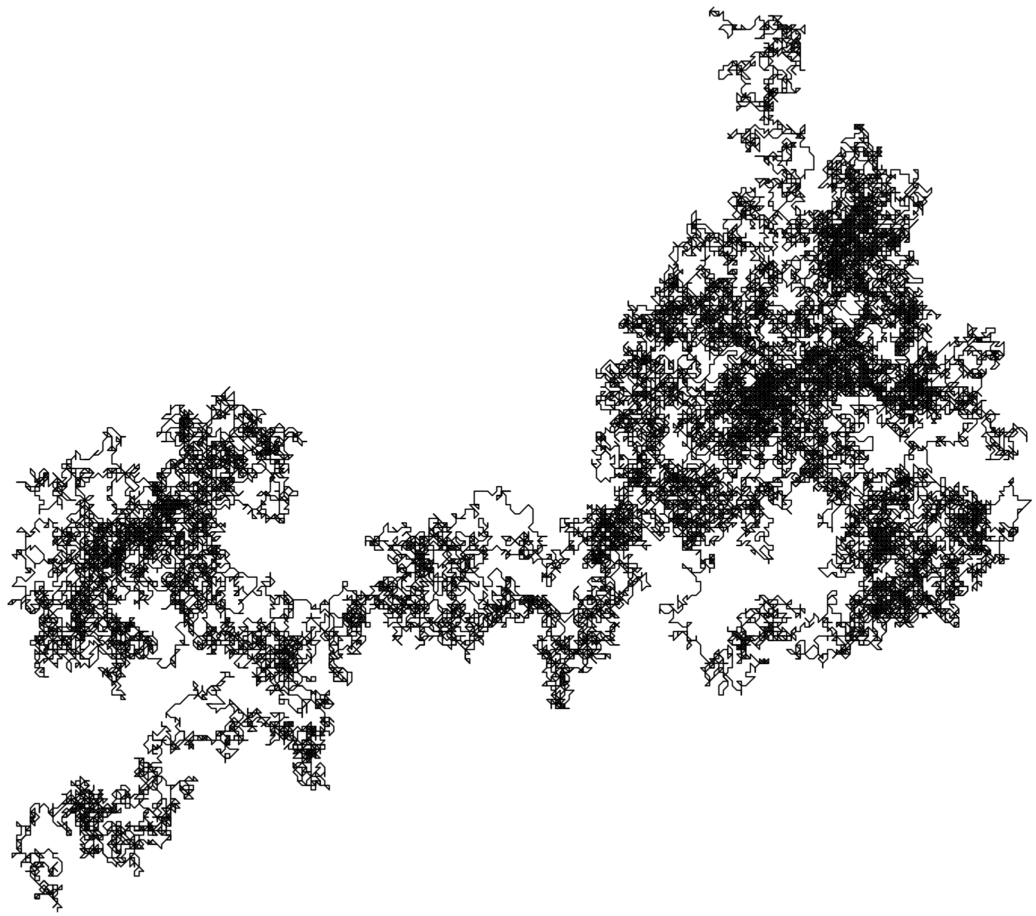
Consider the scaled integer lattice $D_\delta := \delta\mathbb{Z}^2 \cap D$ and let z, $w \in \partial D_\delta$.

Let S be a SRW from z to w in D_δ with loop-erasure Λ and let ω be a SAW from z to w in D_δ.

As $\delta \to 0$:

- SRW: It follows from Donsker's theorem that S converges to Brownian motion connecting distinct boundary points of D (a so-called Brownian excursion). In the case of arbitrary Jordan boundary (say, fractal) this isn't quite true, but a precise statement of the convergence of SRW excursion measure to BM excursion measure was recently established (Kozdron 2006).
- LERW: L-S-W proved that Λ converges to (radial) SLE with $\kappa = 2$.
- SAW: L-S-W proved that if the scaling limit exists and is conformally invariant, then it must be SLE with $\kappa = \frac{8}{3}$.

Frontier or Coastline of a Brownian Island



frontier: boundary of unbounded component of the complement

Figure courtesy Lawler–Schramm–Werner 2003

Mandelbrot's 1982 Conjecture

In *The Fractal Geometry of Nature*, Mandelbrot was looking at (crude) pictures of Brownian islands. He thought that the frontier of the Brownian island looked a lot like the simulations of self-avoiding walk that he had made previously. Since the Hausdorff dimension of the SAW was conjectured to be $\frac{1}{\nu} = \frac{4}{3}$, Mandelbrot made the conjecture that the Hausdorff dimension of the Brownian frontier was also $\frac{4}{3}$.

Theorem: (Lawler–Schramm–Werner 2001)

With probability one, the Hausdorff dimension of the frontier of $B[0,1]$ is $\frac{4}{3}$.

(Non-) Intersections of Brownian Motions

Let B^1 and B^2 be independent Brownian motions with uniformly distributed starting points on $\partial\mathbb{D}$ and stopped when first reaching the circle of radius R.

It follows that there exist exponents $\xi(1,1)$ and $\xi(2,0)$ such that

- $\mathbf{P}(B^1 \cap B^2 = \emptyset) \approx R^{-\xi(1,1)}$
- $\mathbf{P}(B^1 \cup B^2$ does not disconnect 0 from circle of radius $R) \approx R^{-\xi(2,0)}$

Notation: $f(R) \approx g(R)$ iff $\log f(R) \sim \log g(R)$

Much work had been done to show that there exists an entire family of intersection exponents $\xi(j,\lambda)$ satisfying a certain recurrence (or cascade) relationship. However, no one was able to calculate them!

In fact, they were even shown to equal the corresponding interesection exponents for random walks. No one could calculate those either!

Question: What do intersections of BM have to do with Mandelbrot's conjecture?

Answer: (L-S-W 2001)

A point $x \in \mathbb{C}$ is in the ε-neighbourhood of a frontier point if the BM (before time 1) reaches the ε-neighbourhood of x, and if the whole path $B[0,1]$ does not disconnect the disc of radius ε around x from ∞. This whole path can be divided in two parts: B until it reaches the circle of radius ε around x, and B after it reaches this circle. Both parts behave roughly like independent Brownian paths and one can show that the probability that x is in the ε-neighbourhood of a frontier point decays like $\varepsilon^{\xi(2,0)}$ as $\varepsilon \to 0$. Earlier estimates of Lawler imply the Hausdorff dimension of the frontier is $2 - \xi(2,0)$.

Intersection exponents for SLE can be computed directly and then related to intersection exponents for Brownian motion.

Theorem: $\xi(1,1) = \frac{5}{4}$ and $\xi(2,0) = \frac{2}{3}$

Intuition: "Two non-intersecting self-avoiding walks act like two intersecting Brownian paths."