An Introduction to Percolation

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Abstract

Percolation was introduced by S. Broadbent and J. Hammersley in 1957 as a model of fluid flow through a disordered, porous medium. While percolation has existed for over half a century as a well-defined mathematical model, and although there is a significant amount of heuristic and experimental evidence for many remarkable phenomenon, percolation has turned out to be much more difficult than expected to analyze rigorously, and it wasn't until 1980 that H. Kesten proved the first spectacular result in percolation theory. Kesten, combined with work of T. Harris from the 1960's, proved the "obvious result" that the critical probability for bond percolation on $\mathbb{Z}^2$ is $1/2$. Recently there has been an increased interest in two-dimensional percolation mainly due to the fact that critical percolation on the triangular lattice is now completely understood thanks to the introduction of the stochastic Loewner evolution (SLE) by O. Schramm and the work of S. Smirnov and W. Werner, among others. In this talk, we will introduce the mathematical model of percolation, and discuss some of the known results for critical percolation on the two-dimensional triangular lattice including Cardy's formula for crossing probabilities, and the convergence of the discrete percolation exploration process to SLE with parameter 6.

Motivation from Statistical Mechanics

Disclaimer: (Essentially) everything we do will be two-dimensional: $\mathbb{C} \cong \mathbb{R}^2$

A number of simple models introduced in statistical mechanics have proven to be notoriously difficult to analyze in a rigorous mathematical way. These include the **self-avoiding walk**, the **Ising model**, and **percolation**.

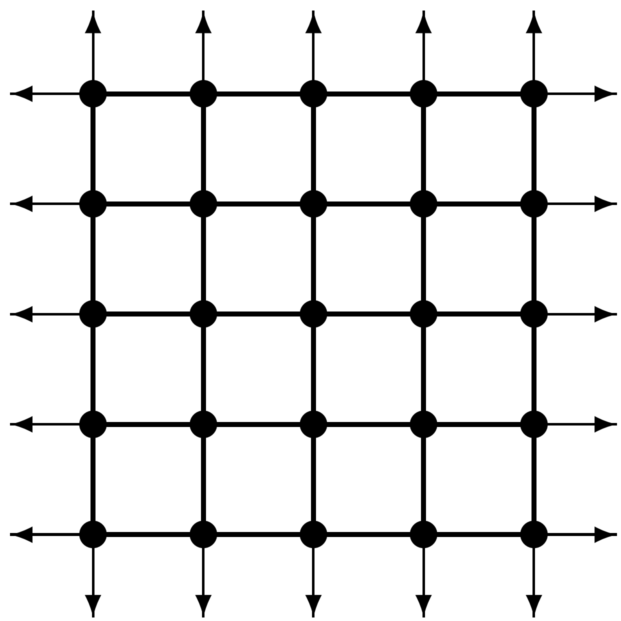
The focus of this talk will be **percolation**.

The Basic Setup

Suppose that Λ is a **graph** consisting of **edges** and **vertices**, and assume that the edges are undirected.

Assume that Λ is connected, infinite, and locally finite (each vertex has finite degree). Later we will assume that all sites of Λ are equivalent; that is, the symmetry group of Λ acts transitively on the vertices.

Think of Λ as lattice-like, e.g., $\Lambda = \mathbb{Z}^2$.



The Basic Setup

For percolation, we say **bonds** instead of edges and **sites** instead of vertices.

Obtain a random subgraph of Λ by selecting bonds/sites independently with the same probability p.

The bonds/sites which are kept are called **open**, otherwise they are called **closed**.

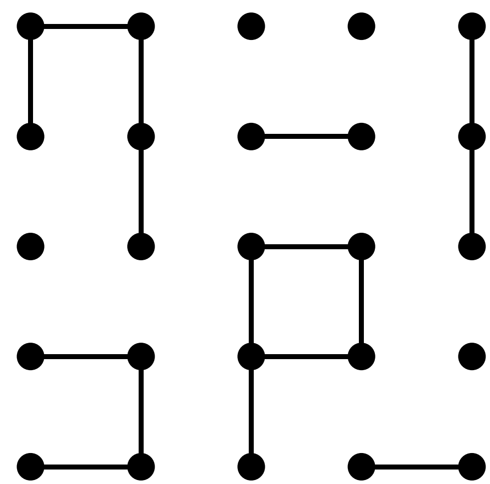
We call the components of the random subgraph the **(open) clusters**.

Let $x \in V(\Lambda)$ be a site. We define C_x to be the open cluster containing x.

If x is not open, then we take $C_x = \emptyset$ (site percolation) or $C_x = \{x\}$ (bond percolation).

Example: Bond Percolation on $\mathbb{Z}^2$

The open subgraph consists of all sites and the open bonds (as indicated by black line segments.)

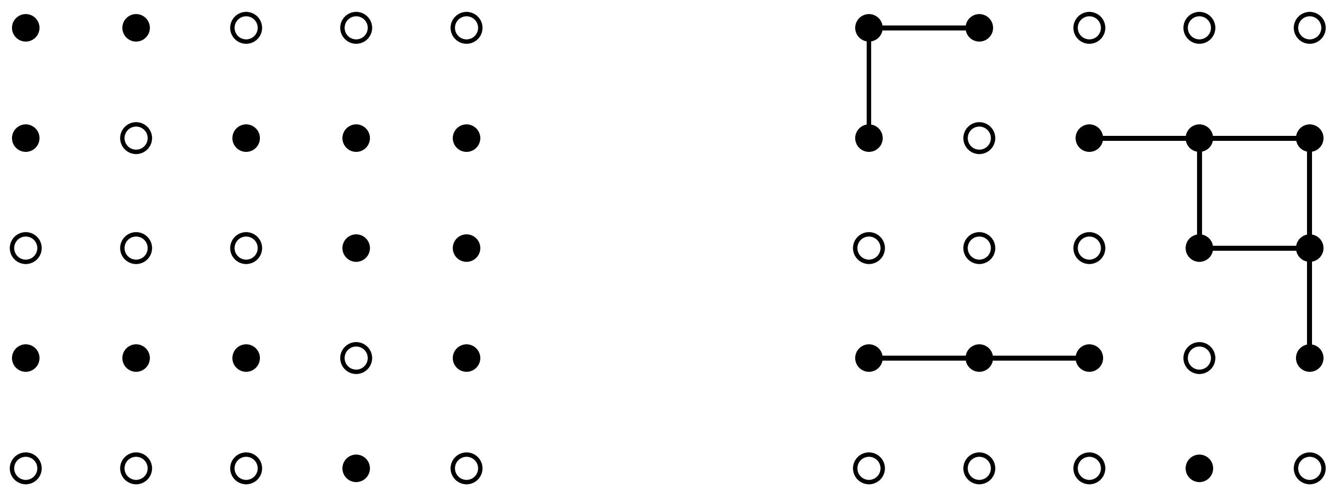


In this picture, there are 10 open clusters.

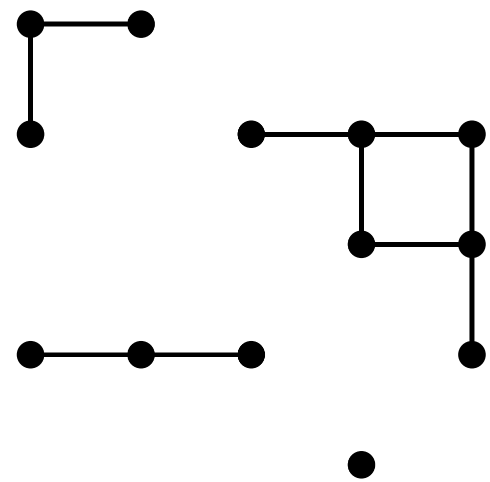
Note that $C_x \neq \emptyset$ for each $x \in V(\mathbb{Z}^2)$.

Example: Site Percolation on $\mathbb{Z}^2$

Begin by selecting the open sites. The open subgraph is then the subgraph of $\mathbb{Z}^2$ induced by these.



Example: Site Percolation on $\mathbb{Z}^2$ (continued)



In this picture, there are 4 open clusters.

Note that there are some $x \in V(\mathbb{Z}^2)$ for which $C_x = \emptyset$.

Note: Visualize water flowing through the open channels. This is why we use the word open and take the bonds to be undirected.

The Percolation Probability

Recall that for a site $x \in V(\Lambda)$ we define C_x to be the open cluster containing x. Let $|C_x| = |V(C_x)|$.

Let $\theta_x(p) = \mathbf{P}_p(|C_x| = \infty) = \mathbf{P}_p(x \leftrightarrow \infty)$.

(Note that $\theta_x(p)$ also depends on the graph and on the type of percolation.)

We now assume that all sites of Λ are equivalent and so $\theta_x(p)$ is the same for all x.

We distinguish $x = 0$ and write $\theta(p) = \theta_0(p)$ and $C = C_0$.

We call $\theta(p)$ the **percolation probability**.

The Critical Probability

If x and y are at a distance d, then $\theta_x(p) \geq p^d \theta_y(p)$.

Therefore, either $\theta_x(p) = 0$ for every site x, or $\theta_x(p) > 0$ for every x.

Furthermore, $\theta(p)$ is increasing in p.

It now follows that there exists a **critical probability** p_c with $0 \leq p_c \leq 1$ such that

- $p < p_c \Longrightarrow \theta(p) = 0$ (i.e., $\theta_x(p) = 0 \ \forall\, x$),
- $p > p_c \Longrightarrow \theta(p) > 0$ (i.e., $\theta_x(p) > 0 \ \forall\, x$).

The Critical Probability

Using Kolmogorov's 0-1 Law, this is equivalent to

- $p < p_c \Longrightarrow \mathbf{P}(\text{there exists an infinite open cluster}) = 0,$
- $p > p_c \Longrightarrow \mathbf{P}(\text{there exists an infinite open cluster}) = 1.$

We say that **percolation occurs** if $\theta(p) > 0$, or equivalently, if $\mathbf{P}(\text{there exists an infinite open cluster}) = 1.$

In other words, water can flow randomly/percolate from 0 to ∞.

Question: What happens at p_c?

Question: What does the graph of p vs. $\theta(p)$ look like (for $p > p_c$)?

The Expected Cluster Size

Let $\chi(p) = \mathbf{E}_p(|C|)$ denote the expected size of the cluster containing the origin.

Clearly $\chi(p) = \infty$ if $p > p_c$.

Question: What happens at p_c?

Question: What does the graph of p vs. $\chi(p)$ look like (for $p < p_c$)?

Critical Exponents for Percolation

Notation: $f(p) \approx g(p)$ as $p \uparrow\downarrow p_c$ means

$$\lim_{p \uparrow\downarrow p_c} \frac{\log f(p)}{\log g(p)} = 1.$$

Definition/Conjecture/Prediction/Open Problem

Critical Probability: As $p \downarrow p_c$,

$$\theta(p) \approx (p - p_c)^{\beta}$$

for some $\beta > 0$. Furthermore, it is believed that $\theta(p_c) = 0$ in general.

Expected Cluster Size: As $p \uparrow p_c$,

$$\chi(p) \approx (p - p_c)^{-\gamma}$$

for some $\gamma > 0$.

Summary of Some Known Results

Bond Percolation on the binary tree

Easy to show $p_c = 1/2$, $\beta = 1$, and $\gamma = 1$.

Site Percolation on $\mathbb{Z}^2$

The value of p_c and the existence of β and γ are still open problems. Numerical simulations show $p_c = 0.592746$.

Bond Percolation on $\mathbb{Z}^2$

Kesten (1980) combined with Harris (1960) showed that $p_c = 1/2$. The existence of β and γ is still an open problem.

Bond Percolation on $\mathbb{Z}^d$, $d \geq 19$

Hara and Slade (1994) proved that $\beta = \gamma = 1$.

Site Percolation on the triangular lattice

(Essentially) everything is known! In particular, $p_c = \frac{1}{2}$ (Kesten and Wierman, 1980s) $\beta = \frac{5}{36}$, and $\gamma = \frac{43}{18}$ (Smirnov and Werner, 2001).

The Conformal Invariance Prediction

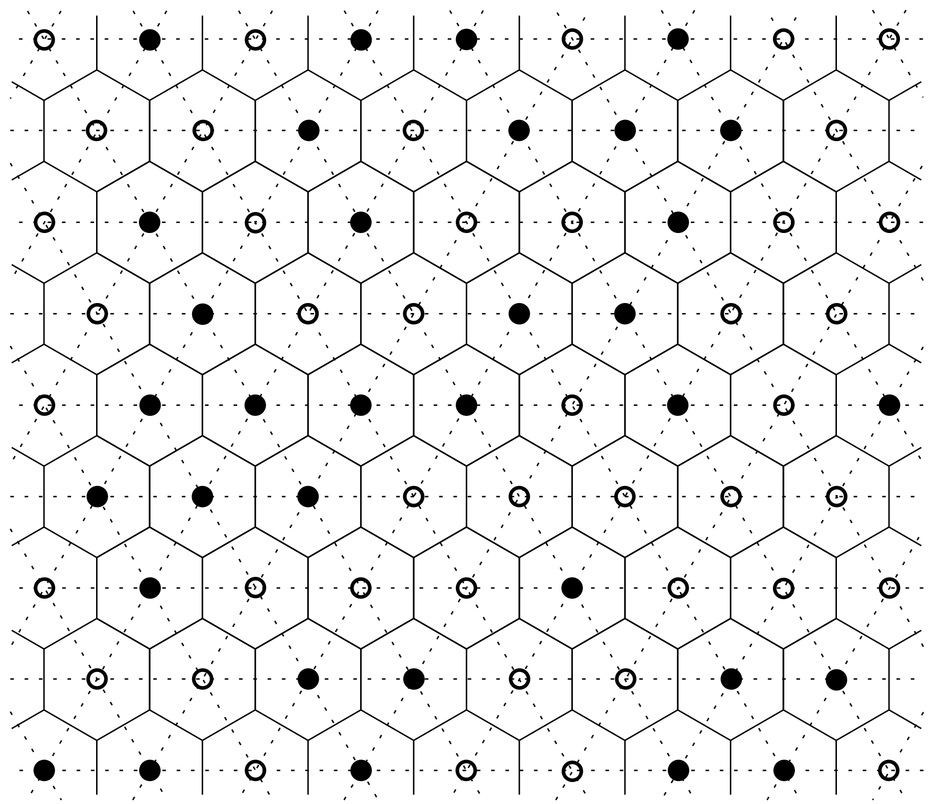
In 1994, Aizenman, Langlands, Pouliot, and Saint-Aubin conjectured, roughly, that if Λ is a planar lattice with suitable symmetry, and we perform critical percolation on Λ, then as the lattice spacing tends to 0, certain limiting probabilities are invariant under conformal transformations.

There is a crude analogy to simple random walk here. Simple random walk on any suitable lattice converges to Brownian motion.

The prediction has only been proved for site percolation on the triangular lattice.

Example: Site Percolation on the Triangular Lattice

Site percolation on the triangular lattice can be identified with "face percolation" on the hexagonal lattice (which is dual to the triangular lattice).



The Discrete Percolation Exploration Path

Consider a simply connected, bounded hexagonal domain with two distinguished external vertices x and y.

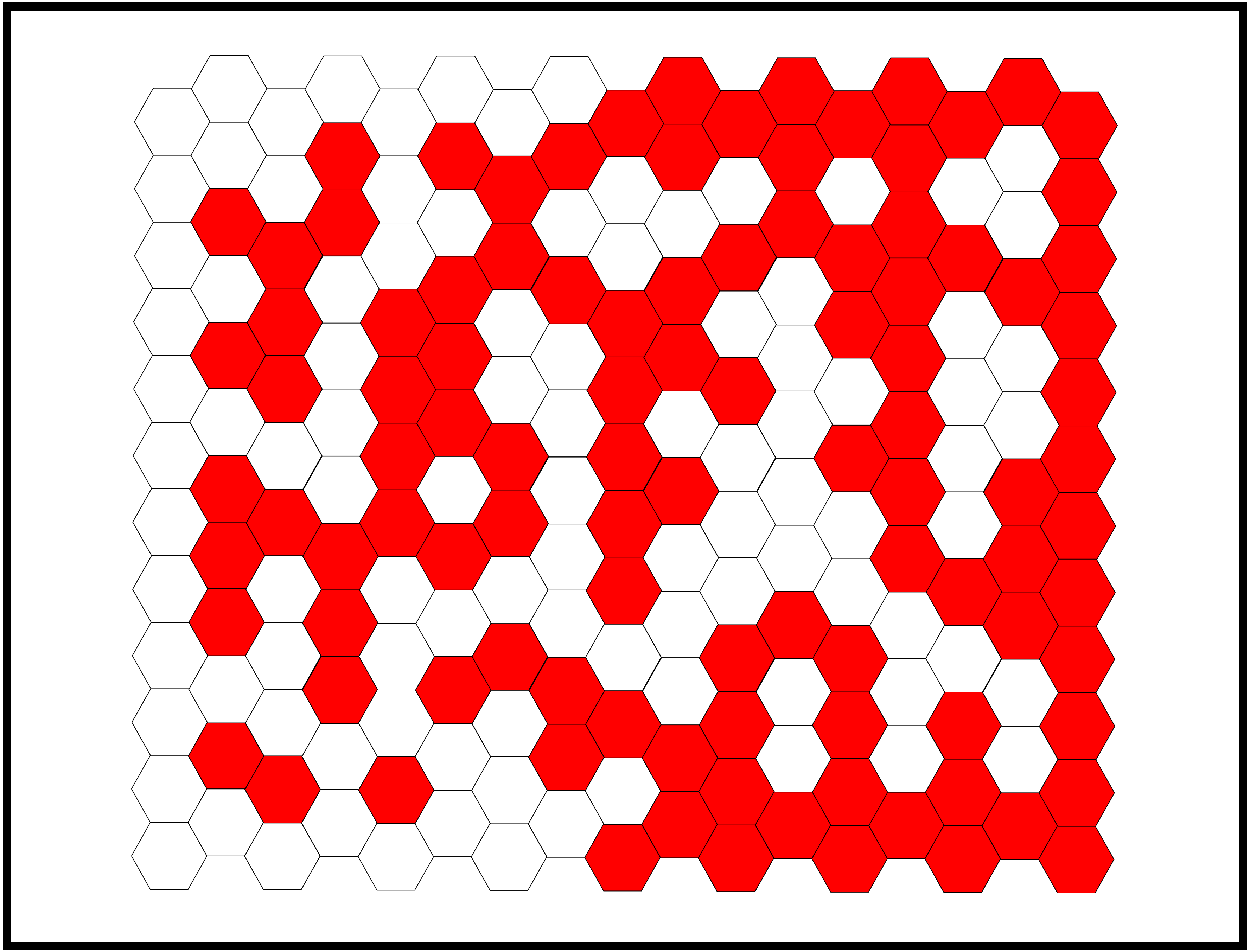
Colour all the hexagons on one half of the boundary from x to y white, and colour all the hexagons on the other half of the boundary from y to x red.

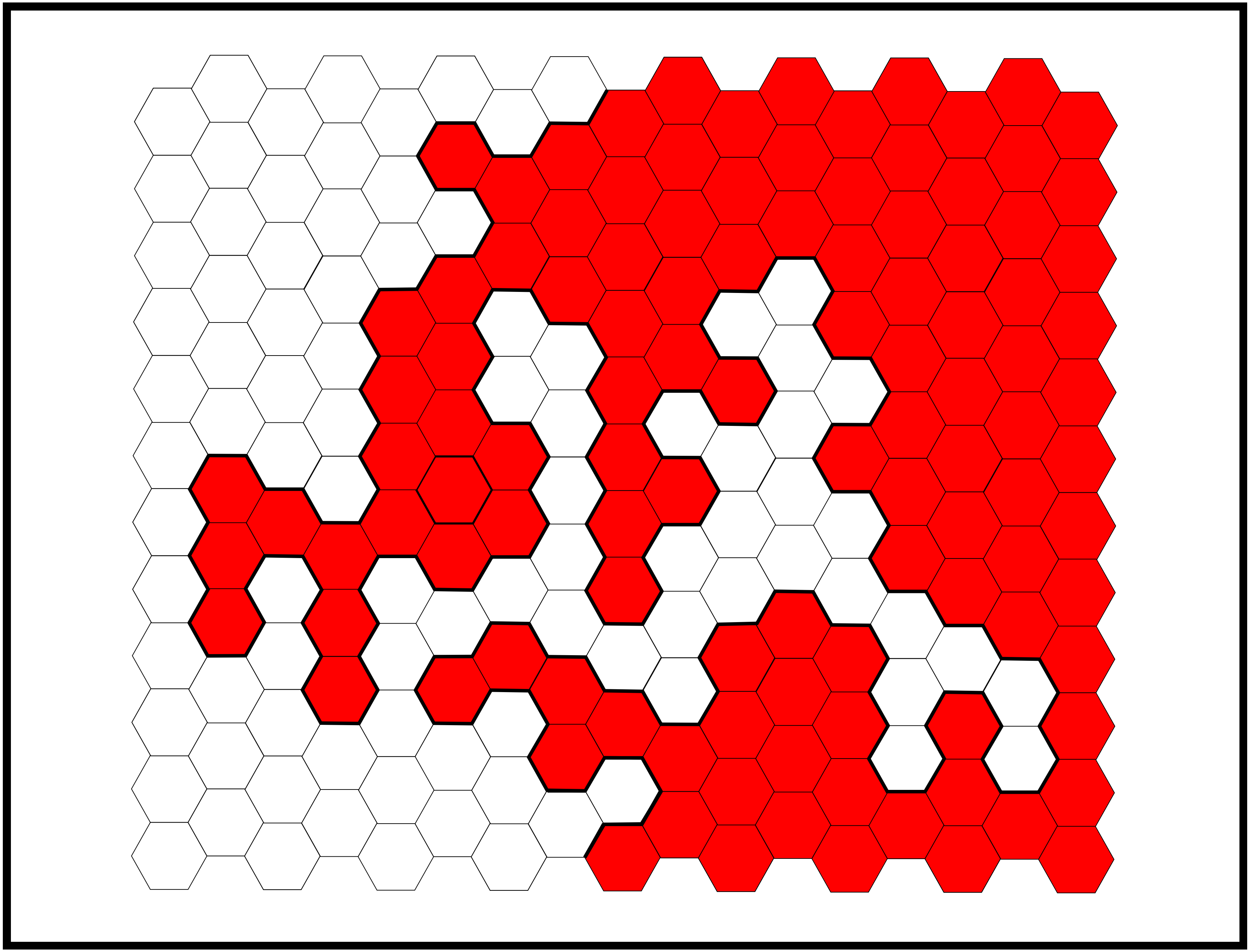
For all remaining interior hexagons colour each hexagon either red or white independently of the others each with probability $1/2$ (i.e., perform critical site percolation on the triangular lattice).

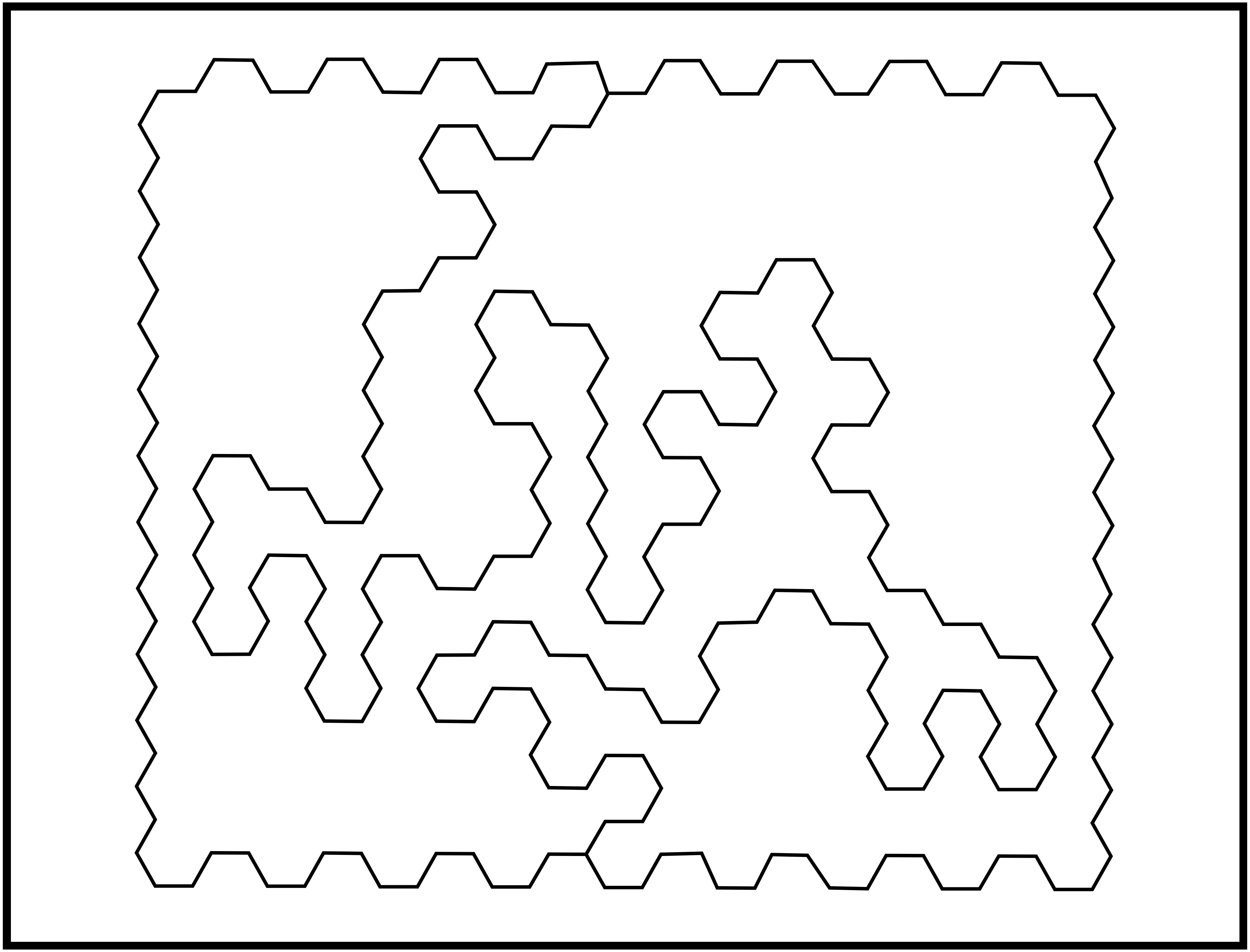
There will be an interface separating the red cluster from the white cluster.

One way is to draw the interface always keeping a red hexagon on the right and a white hexagon on the left.

Another way to visualize the interface is to swallow any islands so that the domain is partitioned into two connected sets.





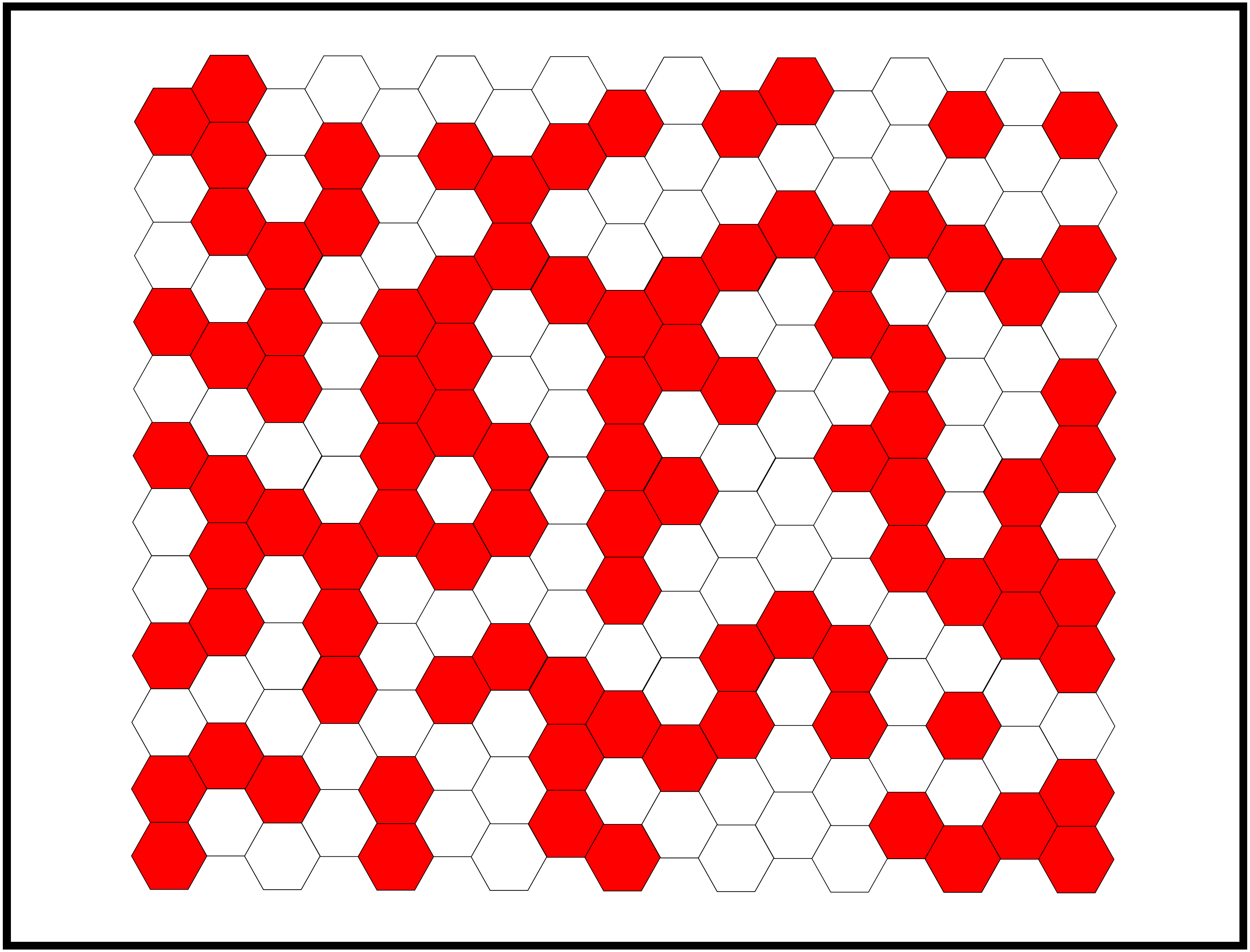


Crossing Probabilities for the Discrete Domain

Consider a simply connected, bounded hexagonal domain D with four distinguished external vertices z_1, z_2, z_3, z_4 ordered counterclockwise. This divides the boundary into four arcs, say A_1, A_2, A_3, A_4.

For all hexagons in D colour each hexagon either red or white independently of the others each with probability 1/2 (i.e., perform critical site percolation on the triangular lattice).

There will necessarily be either a red (open) crossing from A_1 to A_3 or a white crossing from A_2 to A_4.



Approximating the Continuous

Suppose that $D \subset \mathbb{C}$ is a simply connected, bounded Jordan domain containing the origin, and let z_1, z_2, z_3, z_4 be four points ordered counterclockwise around ∂D.

This divides ∂D into 4 arcs, say A_1, A_2, A_3, A_4.

Overlay a suitable lattice with spacing δ over D and consider the resulting lattice-domain D^δ. Identity the original arcs with lattice-domain arcs $A_1^\delta, A_2^\delta, A_3^\delta, A_4^\delta$.

Perform critical percolation on D^δ.

Goal: To understand what happens as $\delta \downarrow 0$?

Question 1: What is the probability that there is a red crossing from A_1^δ to A_3^δ? Call this $P(D;\delta) = P(D, z_1, z_2, z_3, z_4; \delta)$.

Question 2: What is the law or distribution of the scaling limit of the discrete interface?

John Cardy's formula

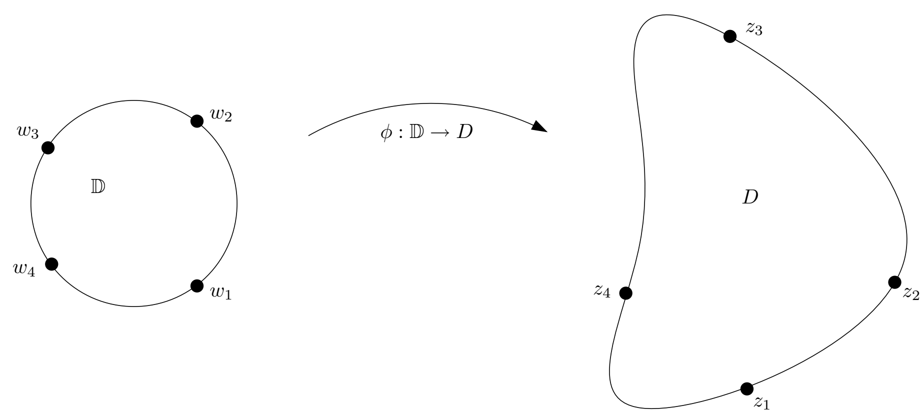
Cardy's Prediction/Formula (1992):

$$\lim_{\delta\to 0} P(D;\delta) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)}\eta^{1/3}{}_2F_1(1/3,2/3;4/3;\eta)$$

where ${}_2F_1$ is the hypergeometric function and

$$\eta = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)}$$

is the cross-ratio with $w_j = \varphi(z_j)$ where $\varphi : \mathbb{D} \to D$ is the unique conformal transformation with $\varphi(0)=0$, $\varphi'(0)>0$.



Lennart Carleson's observation

Using properties of the hypergeometric function one can write

$$\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} z^{1/3} {}_2F_1(1/3, 2/3; 4/3; z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

Furthermore, the function

$$z \mapsto \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

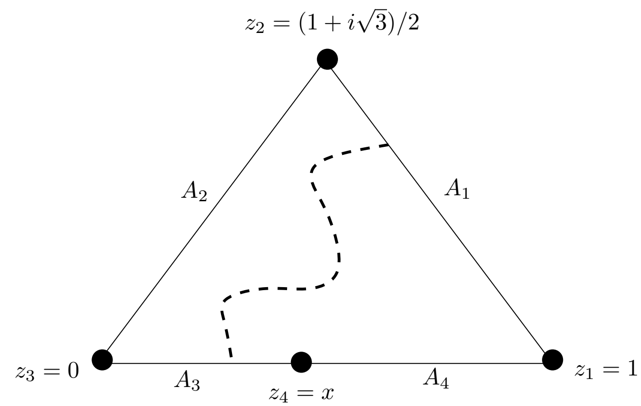
is the Schwarz-Christoffel transformation of the upper half plane onto the equilateral traingle with vertices at 0, 1, and $(1+i\sqrt{3})/2$.

Lennart Carleson's observation

Hence, if D is this equilateral triangle, then Cardy's prediction takes the particularly nice form

$$\lim_{\delta \to 0} P(D; \delta) = x \qquad (*)$$

where x is the following:



Theorem: (Smirnov 2001) Cardy's prediction holds for critical site percolation on the trianglular lattice. Smirnov proved $(*)$ and conformal invariance gave it for all Jordan domains D.

The Scaling Limit of the Exploration Process

Thanks to the work of Smirnov and Werner, there is now a precise description of the scaling limit of the interface (i.e., the exploration process).

Suppose that (D, a, b) is a Jordan domain with distinguished boundary points a and b.

Let $(D^\delta, a^\delta, b^\delta)$ be a sequence of hexagonal lattice-domains with spacing δ which approximate (D, a, b).

(Technically, $(D^\delta, a^\delta, b^\delta)$ converges in the Carathéodory sense to (D, a, b) as $\delta \downarrow 0$.)

Let $\gamma^\delta = \gamma^\delta(D^\delta, a^\delta, b^\delta)$ denote the spacing δ exploration path.

As $\delta \downarrow 0$, the sequence γ^δ converges in distribution to SLE_6 in D from a to b.

What is SLE?

The stochastic Loewner evolution with parameter $\kappa \geq 0$ is a one-parameter family of random conformally invariant curves in the complex plane $\mathbb{C}$ invented by Schramm in 1999.

Let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a simple curve (no self intersections) with $\gamma(0) = 0$, $\gamma(0, \infty) \subseteq \mathbb{H}$, and $\gamma(t) \to \infty$ as $t \to \infty$. e.g., no "spirals"

For each $t \geq 0$ let $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$ be the slit half plane and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map.

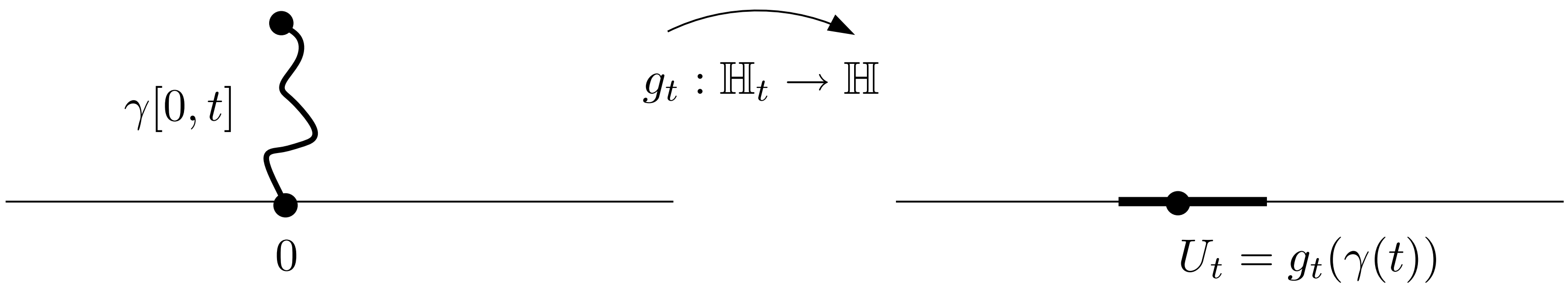
We normalize g_t and parametrize γ in such a way that as $z \to \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

Theorem: (Loewner 1923)

For fixed z, $g_t(z)$ is the solution of the IVP

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$



- The curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0, t]$, $g_t : \mathbb{H}_t \to \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0, t]) \subseteq \mathbb{R}$.

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

The natural thing to do is to start with U_t and solve the Loewner equation.

Solving the Loewner equation gives g_t which conformally map $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$.

Schramm's idea: let U_t be a Brownian motion!

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion.

Definition: SLE$_\kappa$ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

It is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. The following theorem establishes this.

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

SLE$_\kappa$ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem: With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding $\mathbb{R}$.

- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit $\mathbb{R}$.

- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Theorem: With probability one, the Hausdorff dimension of the SLE_κ trace is

$$\min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$