Using SLE to explain a certain observable in the 2d Ising model

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SLE-CFT

The Schramm-Loewner evolution with parameter κ (SLE$_\kappa$) was introduced in 1999 by Oded Schramm while considering possible scaling limits of loop-erased random walk.

Since then, it has successfully been used to study a number of lattice models from two-dimensional statistical mechanics including percolation, uniform spanning trees, self-avoiding walk, and the Ising model.

In general, there is understanding of how SLE can be used to formalize two-dimensional conformal field theory, but nevertheless there is still a lot of work to be done.

In particular, the links between SLE and two-dimensional turbulence, spin glasses, and quantum gravity are currently being investigated.

The goal of this talk is much more modest, namely to explain how a certain non-local observable of the 2d critical Ising model studied by L.-P. Arguin and Y. Saint-Aubin can be rigorously described using SLE$_3$.

SLE-CFT

Conformal field theory (CFT) relies on the concept of a local field and its correlations in order to generate predictions about the model under consideration.

Briefly, in CFT, the central charge $\mathbf{c}$ plays a key role in delimiting the universality classes of a variety of lattice model scaling limits.

We now know that the SLE parameter κ and the central charge $\mathbf{c}$ are related through

$$\mathbf{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

The Ising Model

The Ising model is, perhaps, the simplest interacting many particle system in statistical mechanics. Although it had its origins in magnetism, it is now of importance in the context of phase transitions.

Suppose that $D \subset \mathbb{C}$ is a bounded, simply connected domain with Jordan boundary.

Consider the discrete approximation given by $D \cap \mathbb{Z}^2$.

Assign to each vertex of the square lattice a spin — either up $(+1)$ or down (-1).

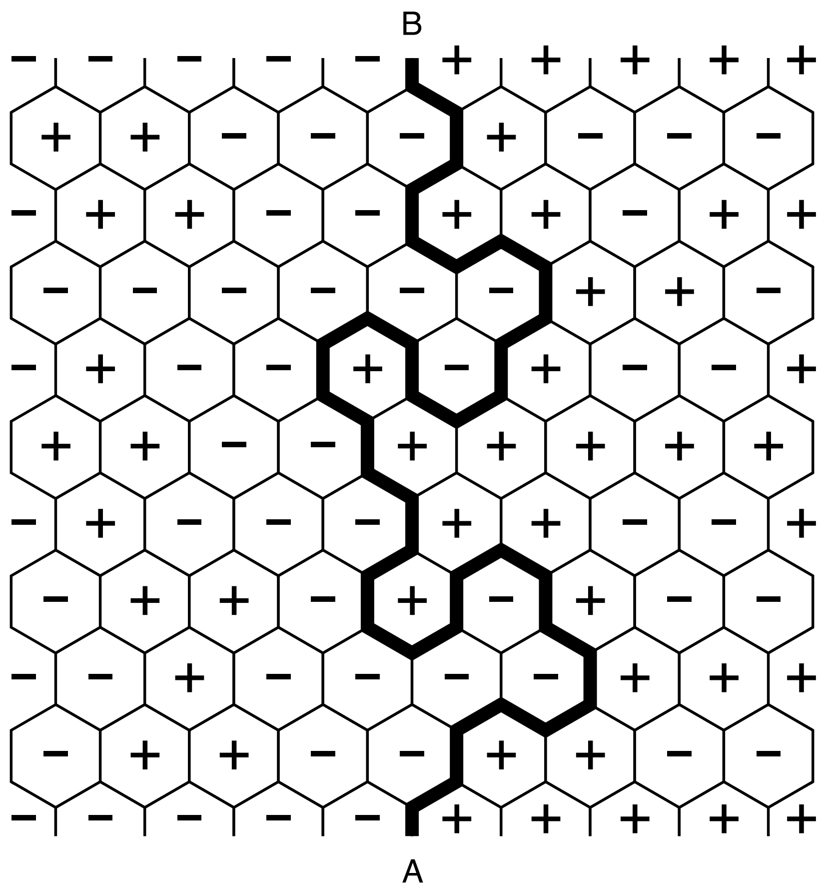
Let ω denote a configuration of spins; i.e., an element of $\Omega = \{-1, +1\}^N$ where N is the number of vertices.

Associate to the configuration the Hamiltonian

$$H(\omega) = -\sum_{i \sim j} \sigma_i \sigma_j$$

where the sum is over all nearest neighbours and $\sigma_i \in \{-1, +1\}$.

The Ising Model



The Ising Model

Define a probability measure

$$P(\{\omega\}) = \frac{\exp\{-\beta H(\omega)\}}{Z}$$

where $\beta > 0$ is a parameter and

$$Z = \sum_{\omega} \exp\{-\beta H(\omega)\}$$

is the partition function (or normalizing constant).

The parameter β is the inverse-temperature $\beta = 1/T$. It is known that there is a critical temperature T_c which separates the ferromagnetic ordered phase (below T_c) from the paramagnetic disordered phase (above T_c).

Furthermore, many physical properties (i.e., observables), such as the thermodynamic free energy, entropy, magnetization, and spin-spin correlation can be determined from the partition function.

The Ising Model

Traditionally, scaling limits in CFT are described by critical exponents.

For example, the spin-spin correlation

$$\langle \sigma_i, \sigma_j \rangle = \sum_{\omega} \sigma_i \sigma_j P(\{\omega\}) \sim \frac{\exp\{-|i-j|/\xi\}}{|i-j|^{\eta}}$$

where the correlation length ξ scales like

$$\xi \sim |T - T_c|^{-\nu}.$$

At T_c, the correlation length ξ diverges, the Ising model becomes scale invariant, and we have

$$\langle \sigma_i, \sigma_j \rangle \sim |i-j|^{-\eta}.$$

The Ising Model

The point-of-view introduced by SLE is that of an interface.

Consider fixing two arcs on the boundary of the domain and holding one boundary arc all at spin up and the other all at spin down.

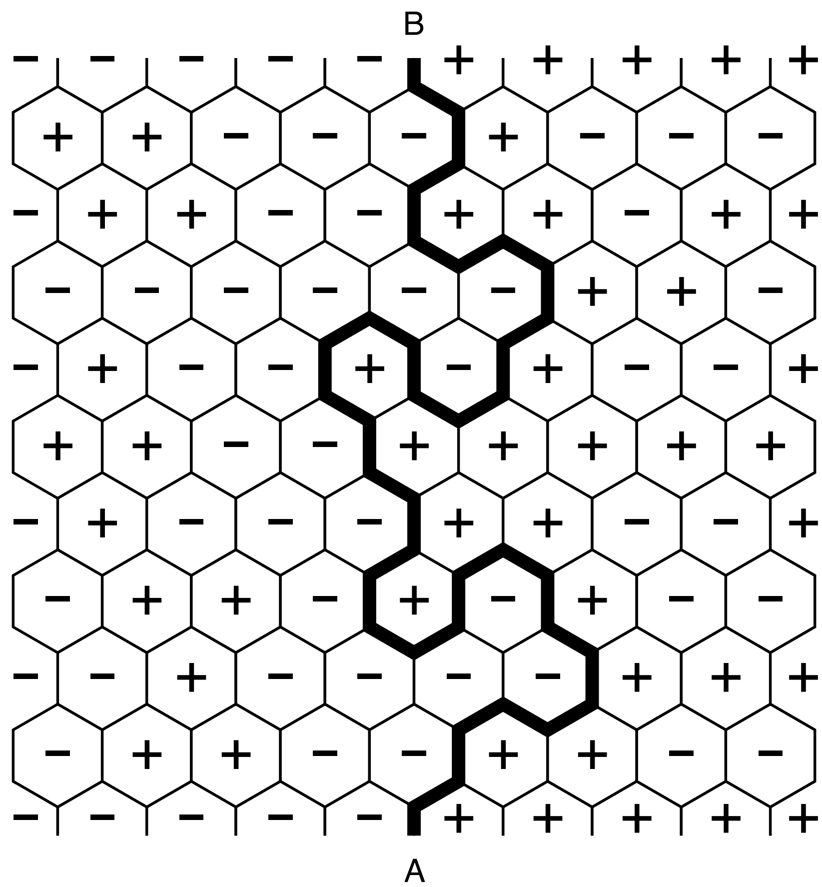
$P(\{\omega\})$ now induces a probability measure on curves (interfaces) connecting the two boundary points where the boundary conditions change.

S. Smirnov has recently showed that as the lattice spacing shrinks to 0, the interfaces converge to SLE$_3$.

Formally, let (D, z, w) be a simply connected Jordan domain with distinguished boundary points z and w. Let $D_n = \frac{1}{n}\mathbb{Z}^2 \cap D$ denote the $1/n$-scale square lattice approximation of D, and let z_n, w_n be the corresponding boundary points of D_n, i.e., we need $(D_n, z_n, w_n) \to (D, z, w)$ in the Caratheodory sense as $n \to \infty$.

If $P_n = P_n(D_n, z_n, w_n)$ denotes the law of the discrete interface, then P_n converges weakly to $\mu_{D,z,w}$, the law of chordal SLE$_3$ in D from z to w.

The Ising Model



Multiple Interfaces in the Ising Model

"Though one can argue whether the scaling limits of interfaces in the Ising model are of physical relevance, their identification opens possibility for computation of correlation functions and other objects of interest in physics." (Smirnov, 2007)

Consider four distinct points z_1, z_2, z_3, z_4 ordered counterclockwise around ∂D. Alternate the boundary conditions between plus and minus, changing at each z_i.

Sample the Ising model at cirticality on D. There will now be two interfaces, either (I) joining $z_1 \leftrightarrow z_4$ and $z_2 \leftrightarrow z_3$, OR (II) joining $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$.

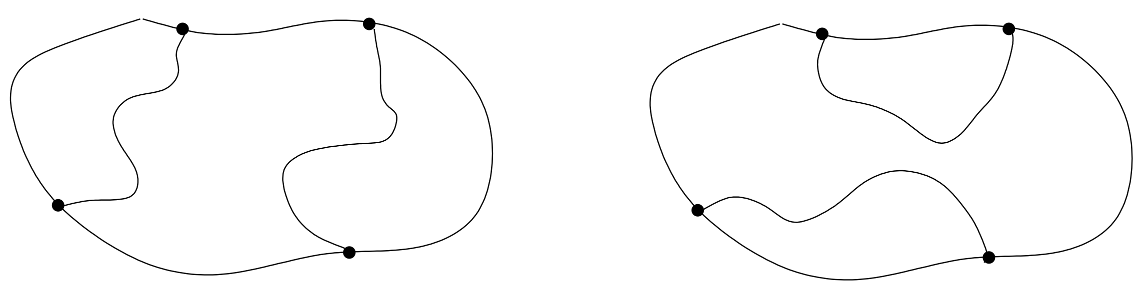


Fig: The two possible configuration-types corresponding to four distinguished boundary points.

Multiple Interfaces in the Ising Model

Question: What is the probability that the resulting crossings are of Type I?

Answer: In the discrete case, it is

$$\frac{Z_I}{Z_I + Z_{II}}$$

where Z_I denotes the partition function corresponding to all possible configurations having a crossing of Type I.

Note: This crossing probability is the non-local observable considered by Arguin and Saint-Aubin.

A Finite Measure on SLE Paths

Let $\mu_D^{\#}(z,w)$ denote the chordal SLE$_\kappa$ probability measure on paths in D from z to w.

Define the finite measure

$$Q_D(z,w) = H_D(z,w)\mu_D^{\#}(z,w)$$

where $H_D(z,w)$ is defined for the upper half plane $\mathbb{H}$ by setting

$$H_{\mathbb{H}}(0,\infty) = 1 \quad \text{and} \quad H_{\mathbb{H}}(x,y) = \frac{1}{|y-x|^{2b}}$$

and for other simply connected domains D by conformal covariance

$$H_D(z,w) = |f'(z)|^b \, |f'(w)|^b H_{D'}(f(z), f(w))$$

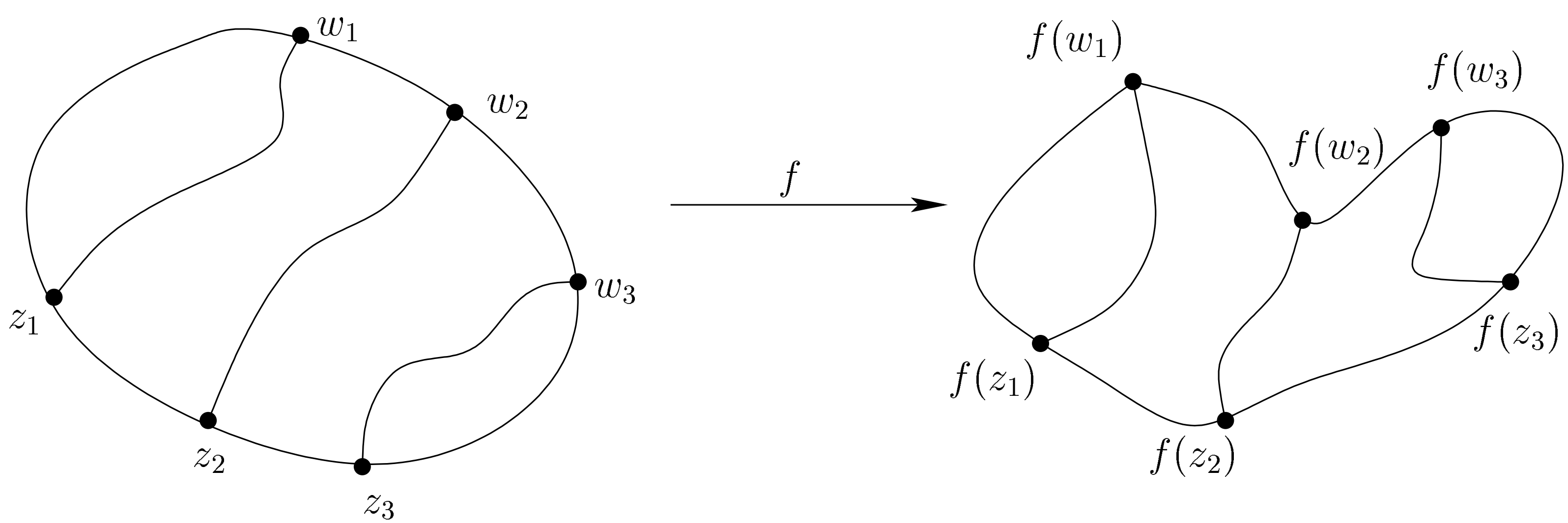
where $f : D \to D'$ is a conformal transformation (assuming appropriate smoothness) and $b > 0$ is a parameter.

A Finite Measure on Multiple SLE Paths

Let $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$ denote n-tuples of distinct points in ∂D ordered counterclockwise.

Goal: To construct a finite measure $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves with γ_i connecting z_i to w_i.

This measure should satisfy conformal covariance, along with certain other properties.



A Finite Measure on Multiple SLE Paths

$Q_{D,b,n}(\mathbf{z},\mathbf{w})$, the n-path SLE$_\kappa$ measure in D, is defined to be the measure that is absolutely continuous with respect to the product measure

$$Q_{D,b}(z_1,w_1)\times\cdots\times Q_{D,b}(z_n,w_n)$$

with Radon-Nikodym derivative $Y(\bar\gamma)=Y_{D,b,\mathbf{z},\mathbf{w}}(\gamma^1,\ldots,\gamma^n)$ given by

$$Y(\bar\gamma)=1\{\gamma^k\cap\gamma^l=\emptyset,\ 1\le k<l\le n\}\ \exp\left\{-\lambda\sum_{k=1}^{n-1}m(D;\gamma^k,\gamma^{k+1})\right\}$$

where $m(D;V_1,V_2)$ is the Brownian loop measure of loops in D intersecting both V_1 and V_2.

If $\lambda\ge -1/2$, it can be shown that $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is a finite measure.

Existence of the Configurational Measure

Theorem (K-Lawler, 2007): *For any $b \geq \frac{1}{4}$, there exists a family of measures $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves satisfying*

- *conformal covariance,*
- *boundary perturbation,*
- *cascade relation,*
- *Markov property.*

Moreover, the simple curve γ^i is a chordal SLE$_\kappa$ from z_i to w_i in D where

$$\kappa = \frac{6}{2b+1} \longleftrightarrow b = \frac{6-\kappa}{2\kappa}.$$

Note: $b \geq \frac{1}{4} \longleftrightarrow 0 < \kappa \leq 4$

Note: These four properties were not discovered accidentally. We were told by CFT what properties the measure had to satisfy, and what the relationship between all the parameters had to be.

Parameters

$$a = \frac{2}{\kappa}, \quad b = \frac{3a-1}{2}, \quad \lambda = \frac{(3a-1)\,(4a-3)}{2a}, \quad \mathbf{c} = -2\lambda, \quad \text{and} \quad d = 1 + \frac{1}{4a}.$$

These parameters have interpretations which can be summarized as follows.

- $\kappa \in (0, 4]$ is the variance of the driving Brownian motion in SLE$_\kappa$ if the half-plane capacity at time t is $2t$.
- $a \in [1/2, \infty)$ and at is the half-plane capacity at time t for SLE$_\kappa$ if the driving Brownian motion is chosen to have variance 1.
- $b \in [1/4, \infty)$ is the boundary scaling exponent or boundary conformal weight.
- $\lambda \in [-1/2, \infty)$ is the exponent of the Brownian loop measure that arises in the Radon-Nikodym derivatives.
- $\mathbf{c} \in (-\infty, 1]$ is the central charge.
- $d \in (1, 3/2]$ is the Hausdorff dimension of the paths.

The Partition Function for Two Paths

Define $H_{D,b,n}(\mathbf{z},\mathbf{w})$ to be the mass of the measure $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ and note that $H_{D,b,n}$ satisfies the scaling rule

$$H_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, H_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})).$$

Here $|f'(\mathbf{z})| = |f'(z_1)| \cdots |f'(z_n)|$.

Furthermore, if we define

$$\tilde{H}_{D,b,n}(\mathbf{z},\mathbf{w}) = \frac{H_{D,b,n}(\mathbf{z},\mathbf{w})}{H_{D,b}(z_1,w_1)\cdots H_{D,b}(z_n,w_n)},$$

then this is a conformal invariant.

The Partition Function for Two Paths

By conformal invariance, it suffices to work in $D = \mathbb{H}$. Let $0 < x < y < \infty$.

Proposition: If $b \geq 1/4$, then

$$\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y)) = \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\,F(2a, 1-2a, 4a; x/y).$$

where $F = {}_2F_1$ denotes the hypergeometric function and $a = \dfrac{2}{\kappa} = \dfrac{2b+1}{3}$.

Note: This result first appeared rigorously in J. Dubédat, and was derived using CFT by M. Bauer, D. Bernard, and K. Kytölä. Our configurational approach provides another rigorous derivation.

Note: As we will see in a moment, the special case of the Ising model actually appeared earlier in L.-P. Arguin and Y. Saint-Aubin.

Note: Although our construction is restricted to simple curves ($0 < \kappa \leq 4$), if we formally plug in $\kappa = 6$, then we recover Cardy's formula for percolation.

The Partition Function for Two Paths

The proof of this proposition is accomplished by deriving and then solving a differential equation satisfied by $\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y))$.

By scaling, $\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y)) = \phi(x/y)$ for some function $\phi = \phi_{\mathbb{H},b}$ of one variable.

It can be shown that the ODE satisfied by ϕ is

$$u^2\,(1-u)^2\,\phi''(u) + 2\,u\,(a-u+(1-a)\,u^2)\,\phi'(u) - a(3a-1)(1-u)^2\,\phi(u) = 0.$$

In the case that $\kappa = 3$ (so that $a = 2/3$), this differential equation reduces to

$$3u^2\,(1-u)\,\phi''(u) + 2\,u\,(2-u)\,\phi'(u) - 2(1-u)\,\phi(u) = 0.$$

Let $g(z) = \phi(1-z)$ so that g satisfies

$$3z(z-1)^2\,g''(z) + 2\,(z-1)\,(z+1)\,g'(z) - 2z\,g(z) = 0.$$

The Partition Function for Two Paths

For the Ising model, note that

$$\kappa = 3, \quad a = \frac{2}{3}, \quad b = \frac{1}{2}, \quad \lambda = -\frac{1}{4}, \quad \mathbf{c} = \frac{1}{2}, \quad d = \frac{11}{8}.$$

Also, recall that

$$3z(z-1)^2\, g''(z) + 2\,(z-1)\,(z+1)\, g'(z) - 2z\, g(z) = 0.$$

This differential equation is exactly the one that was derived by L.-P. Arguin and Y. Saint-Aubin in 2002 using techniques from conformal field theory in order to obtain theoretical predictions for the behaviour of the crossing probability (i.e., the non-local observable for the 2-D Ising model.)

For Arguin and Saint-Aubin, the function g was, basically, the "four-point correlation function of the local field of conformal weight $1/2$."

Calculating the Crossing Probability

By conformal invariance, it is enough to work in the upper half plane $\mathbb{H}$, with boundary points 0, 1, ∞, and x where $0 < x < 1$ is a real number.

The possible interface configurations are therefore of two types, namely (I) a simple curve connecting 0 to ∞ and a simple curve connecting x to 1, or (II) a simple curve connecting 0 to x and a simple curve connecting 1 to ∞.

The configurational measure corresponding to Type I is

$$Q_{\mathbb{H},b,2}((0,x),(1,\infty))$$

and the configurational measure corresponding to Type II is

$$Q_{\mathbb{H},b,2}((x,1),(\infty,0)).$$

By symmetry, however,

$$Q_{\mathbb{H},b,2}((x,1),(\infty,0)) = Q_{\mathbb{H},b,2}((0,1-x),(1,\infty)).$$

Calculating the Crossing Probability

Therefore, the partition function corresponding to Type I is (defined as)

$$Z_{b,I}(x) := H_{\mathbb{H},b,2}((0,x),(1,\infty))$$

and the partition function corresponding to Type II is

$$Z_{b,II}(x) := H_{\mathbb{H},b,2}((0,1-x),(1,\infty)).$$

Finally, we conclude that the probability of a crossing of Type I is

$$\frac{Z_{b,I}(x)}{Z_{b,I}(x) + Z_{b,II}(x)}.$$

In the case of the Ising model, $b = 1/2$, this recovers the results of Arguin and Saint-Aubin.