Investigating the Hitting Distribution of the Unit Circle by Chordal SLE in the Upper Half Plane

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Schramm-Loewner Evolution

It is well-known that the Schramm-Loewner evolution (SLE) is a powerful tool for studying two-dimensional statistical mechanics models at criticality.

In fact, SLE results have given new insights into many lattice models including percolation, the Ising model and related random cluster models, uniform spanning trees, self-avoiding walk, and the Gaussian free field.

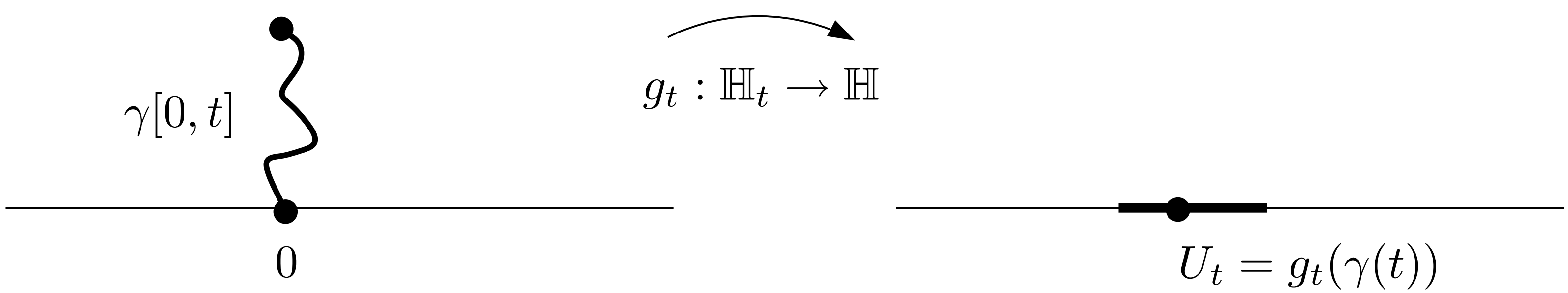
However, it is also interesting to study properties of SLE as a stochastic process.

While several results are known about path properties of SLE, there are still many unasked questions.

For example, when we study stochastic processes, it is natural to study first hitting times and exit distributions.

Compare to the *Handbook of Brownian Motion – Facts and Formulae* by Andrei N. Borodin and Paavo Salminen which contains 685 pages and over 2500 formulas.

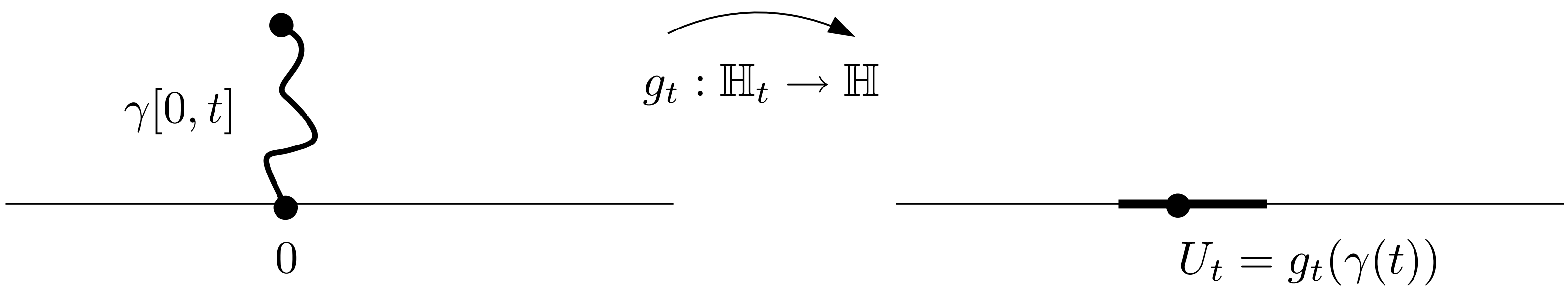
The Picture That Says It All!



- The simple curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- The curve γ never re-visits $\mathbb{R}$; that is, $\gamma(0,t) \subset \mathbb{H}$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0,t]$ denotes the slit plane.
- $g_t : \mathbb{H}_t \to \mathbb{H}$ is a conformal map;
- $U_t := g_t(\gamma(t))$ is the unique point on $\mathbb{R}$ that is the image of the tip, $\gamma(t)$.
- $t \mapsto U_t$ is continuous.

What is SLE?

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by the Loewner equation.



We (uniquely) normalize g_t and (re-)parametrize γ in such a way that as $z \to \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(|z|^{-2}\right).$$

Theorem. (Loewner 1923)
If $z \in \mathbb{H}$ with $z \notin \gamma[0,\infty]$, then the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the IVP

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Schramm-Loewner Evolution

The natural thing to do is to start with a continuous function $t \mapsto U_t$ and solve the Loewner equation.

Solving the Loewner equation gives g_t which conformally maps $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$ and $K_t = \gamma(0, t]$.

While studying loop-erased random walk, Schramm's idea was to let U_t be a Brownian motion! (In retrospect, it is natural.)

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion.

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

Definition. SLE_κ in the upper half plane is the random collection of conformal maps $g_t = g(t,\cdot)$ obtained by solving the Loewner equation

$$\frac{\partial}{\partial t} g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa} B_t}, \quad g(0,z) = z.$$

It is not obvious that $g^{-1}(t,\cdot)$ is well-defined at U_t so that the curve γ can be defined. A deep theorem due to Rohde and Schramm proves this is true.

Think of $\gamma(t) = g^{-1}(t, \sqrt{\kappa} B_t)$.

SLE_κ is the random collection of conformal maps $g(t,\cdot)$ (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem. (Rohde-Schramm 2001; Lawler-Schramm-Werner 2004)

With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding $\mathbb{R}$.

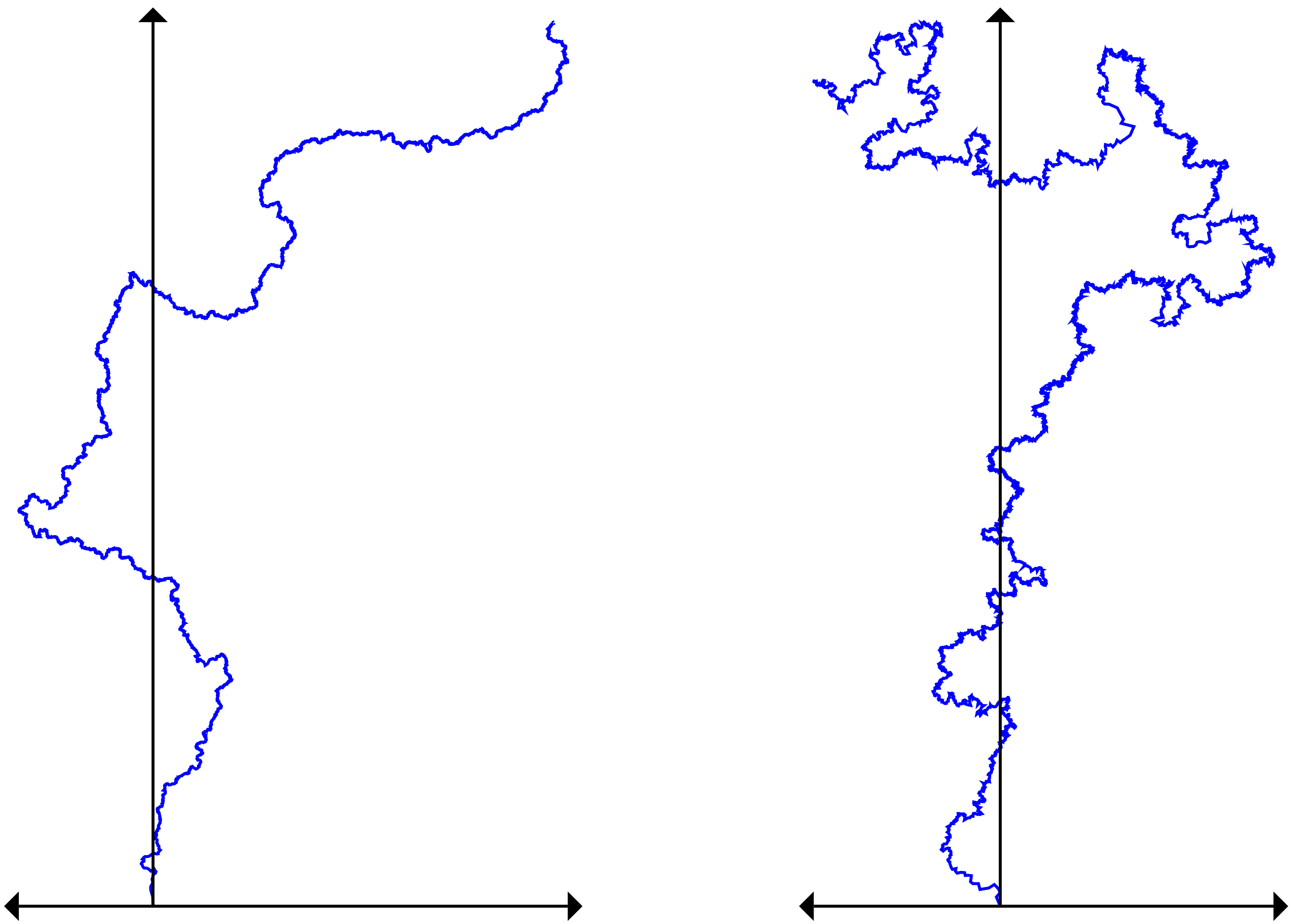
- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit $\mathbb{R}$.

- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

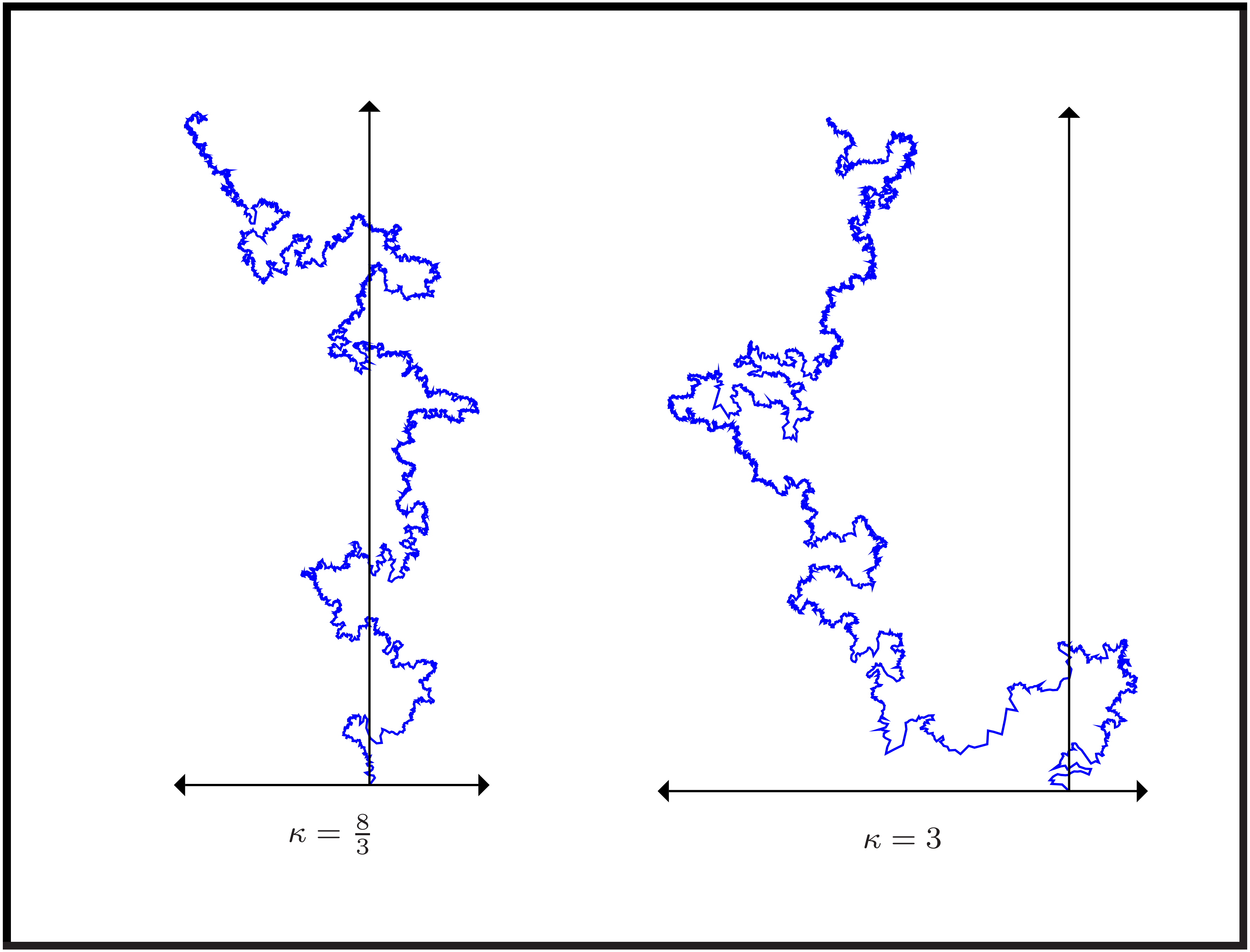
Theorem. (Beffara 2004, 2008)

With probability one, the Hausdorff dimension of the SLE$_\kappa$ trace is

$$\min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$



$\kappa = 1$ $\kappa = 2$


$\kappa = \frac{8}{3}$
$\kappa = 3$

Goal: To study the hitting distribution of the first visit to the circle of radius 1 by chordal SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$.

Let $\gamma_\kappa : [0,\infty) \to \overline{\mathbb{H}}$ be a chordal SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$.

- $\dfrac{\partial}{\partial t} g_\kappa(t,z) = \dfrac{2}{g_\kappa(t,z) - \sqrt{\kappa}\, B(t)}, \quad g_\kappa(0,z) = z,$
- $g_\kappa(t, \gamma_\kappa(t)) = \sqrt{\kappa}\, B(t) \sim B(\kappa t)$

Let $\tau_\kappa = \inf\{t > 0 : |\gamma_\kappa(t)| = 1\}$ denote the first time that the SLE$_\kappa$ trace hits the disk of radius 1 centred at the origin.

Let $\Theta_\kappa = \arg(\gamma_\kappa(\tau_\kappa))$ denote the argument of $\gamma_\kappa(\tau_\kappa)$.

Note that the symmetry of SLE about the imaginary axis immediately implies that

$$P\{\Theta_\kappa \in [0, \pi/2]\} = P\{\Theta_\kappa \in [\pi/2, \pi]\} = \frac{1}{2}.$$

The $\kappa = 0$ degenerate case

If $\kappa = 0$, then the chordal Loewner equation is

$$\frac{\partial}{\partial t} g_0(t,z) = \frac{2}{g_0(t,z)}, \quad g_0(0,z) = z,$$

which implies that

$$g_0(t,z) = \sqrt{z^2 + 4t}.$$

Since $\gamma_0(t)$ satisfies $g_0(t, \gamma_0(t)) = 0$ for all $t \geq 0$ we conclude that

$$\gamma_0(t) = 2i\sqrt{t}$$

so that Θ_0 is exactly equal to $\pi/2$.

The $\kappa \to \infty$ degenerate case

The basic idea is that as $\kappa \to \infty$, the diameter of the SLE trace is large although the amount of time needed for the diameter to become large is small.

That is, $\Im\gamma_\kappa(t) \to 0$ as $\kappa \to \infty$ so that

$$\Theta_\infty \in \{0, \pi\}$$

each with probability $1/2$.

$\kappa = 6$: *locality and an exact calculation*

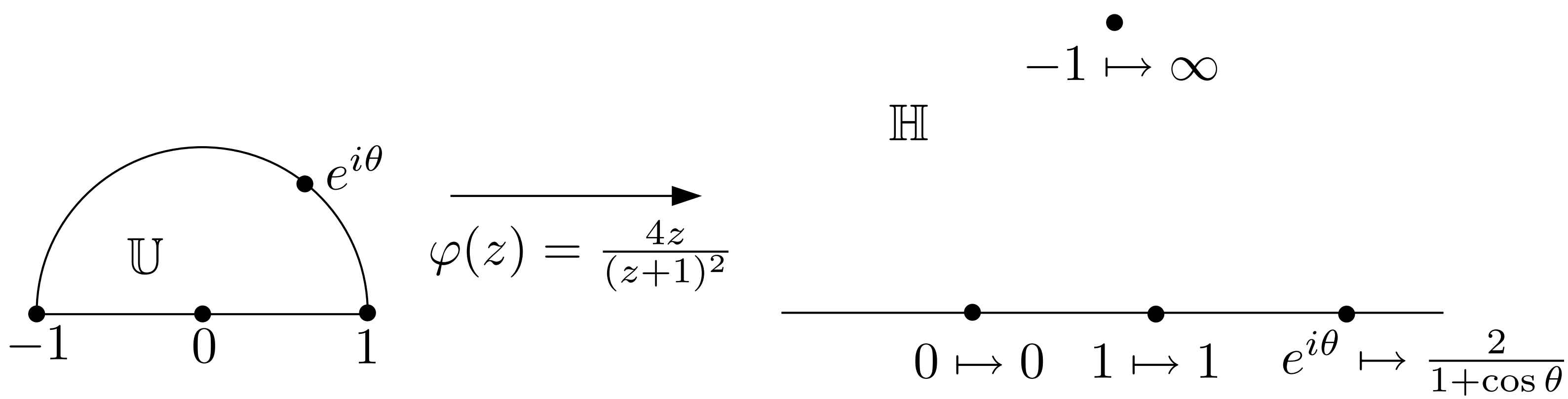
Observe that for $0 \leq \theta < \pi$, the event that

$$\{\Theta_6 \in [\theta, \pi]\}$$

is the same as the event that there is a "crossing" from the interval $[0, 1]$ to the (counterclockwise) arc $[e^{i\theta}, e^{i\pi}]$ in the domain $\mathbb{U} = \mathbb{D} \cap \mathbb{H}$.

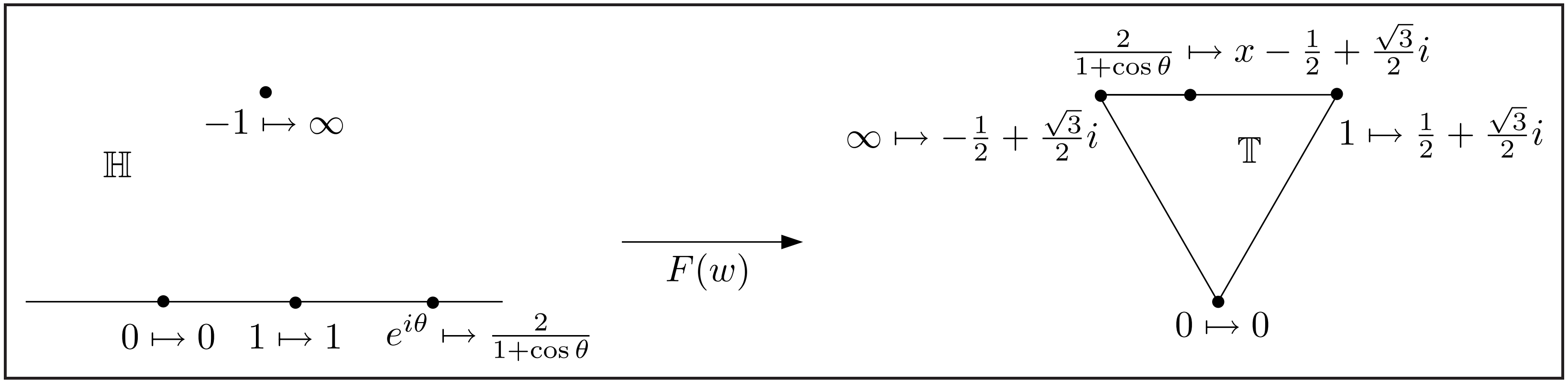
The locality property of SLE_6 implies that the probability of this event is given by Cardy's formula.

Let $\mathbb{U} = \mathbb{D} \cap \mathbb{H}$ denote the upper half disk and consider the conformal transformation $\varphi : \mathbb{U} \to \mathbb{H}$.



Let $\mathbb{T}$ denote the equilateral triangle with vertices at 0, $e^{i\pi/3}$, and $e^{2i\pi/3}$ and consider the Schwarz-Christoffel transformation of the upper half-plane onto $\mathbb{T}$ given by

$$F(w) = -\frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^w \zeta^{-2/3}(\zeta - 1)^{-2/3} d\zeta.$$



After several lines of calculations, it can be shown that

$$x = 1 - \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_1^{\frac{2}{1+\cos\theta}} \zeta^{-2/3}(\zeta - 1)^{-2/3} d\zeta \in (0, 1).$$

Stated in terms of the equilateral triangle $\mathbb{T}$, Cardy's formula takes the form

$$P\Big\{\exists \text{ crossing from the side } [0, e^{i\pi/3}] \text{ to the line segment } \Big[-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, x-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\Big] \text{ in } \mathbb{T}\Big\} = x.$$

The conformal invariance of SLE implies that

$$P\Big\{\exists \text{ crossing from the the interval } [0,1] \text{ to the arc } [e^{i\theta}, e^{i\pi}] \text{ in } \mathbb{U}.\Big\}$$
$$= P\Big\{\exists \text{ crossing from the side } [0, e^{i\pi/3}] \text{ to the line segment } \Big[-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, x-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\Big] \text{ in } \mathbb{T}\Big\}$$

and so we conclude that for $0 \le \theta \le \pi$,

$$P\{\Theta_6 \le \theta\} = 1 - x = \frac{\Gamma(2/3)}{\Gamma(1/3)^2}\int_1^{\frac{2}{1+\cos\theta}} \zeta^{-2/3}(\zeta-1)^{-2/3}d\zeta.$$

Making the change-of-variables

$$z = \frac{\zeta - 1}{\zeta}, \quad \zeta = \frac{1}{1-z}, \quad \zeta - 1 = \frac{z}{1-z}, \quad d\zeta = \frac{dz}{(1-z)^2}$$

implies that

$$P\{\Theta_6 \le \theta\} = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^{\frac{1-\cos\theta}{2}} z^{-2/3}(1-z)^{-2/3} dz.$$

If the random variable X is defined by

$$X = \frac{1 - \cos(\Theta_6)}{2},$$

then the distribution of X is given by

$$\begin{aligned} P\{X \le x\} = P\left\{\frac{1-\cos(\Theta_6)}{2} \le x\right\} &= P\{\Theta_6 \le \cos^{-1}(1-2x)\} \\ &= \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^x z^{-2/3}(1-z)^{-2/3} dz. \end{aligned}$$

In other words, $X \sim \beta(1/3, 1/3)$.

This suggests that the random variable

$$X_\kappa = \frac{1 - \cos(\Theta_\kappa)}{2}$$

might be the one to study. In fact, there might be some function $\alpha = \alpha(\kappa)$ such that

$$X_\kappa \sim \beta(\alpha, \alpha).$$

The corresponding density is

$$f(x) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\, x^{\alpha-1}(1-x)^{\alpha-1}$$

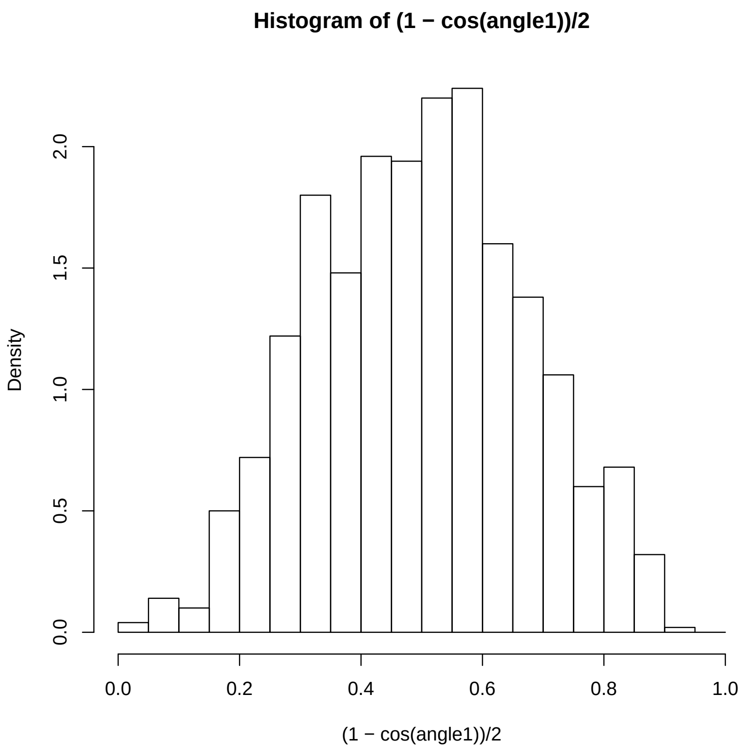
for $0 \leq x \leq 1$.

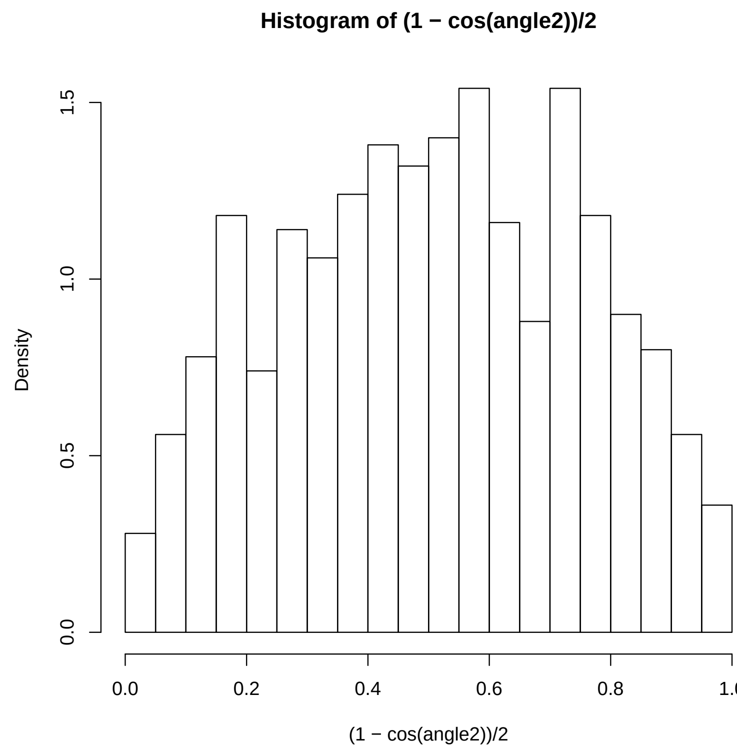
- $X_0 \sim \beta(\infty, \infty)$ (As $\alpha \to \infty$, the support of the density converges to $\{1/2\}$.)
- $X_6 \sim \beta(1/3, 1/3)$
- $X_\infty \sim \beta(0, 0)$ (As $\alpha \to 0+$, the support of the density converges to $\{0, 1\}$.)

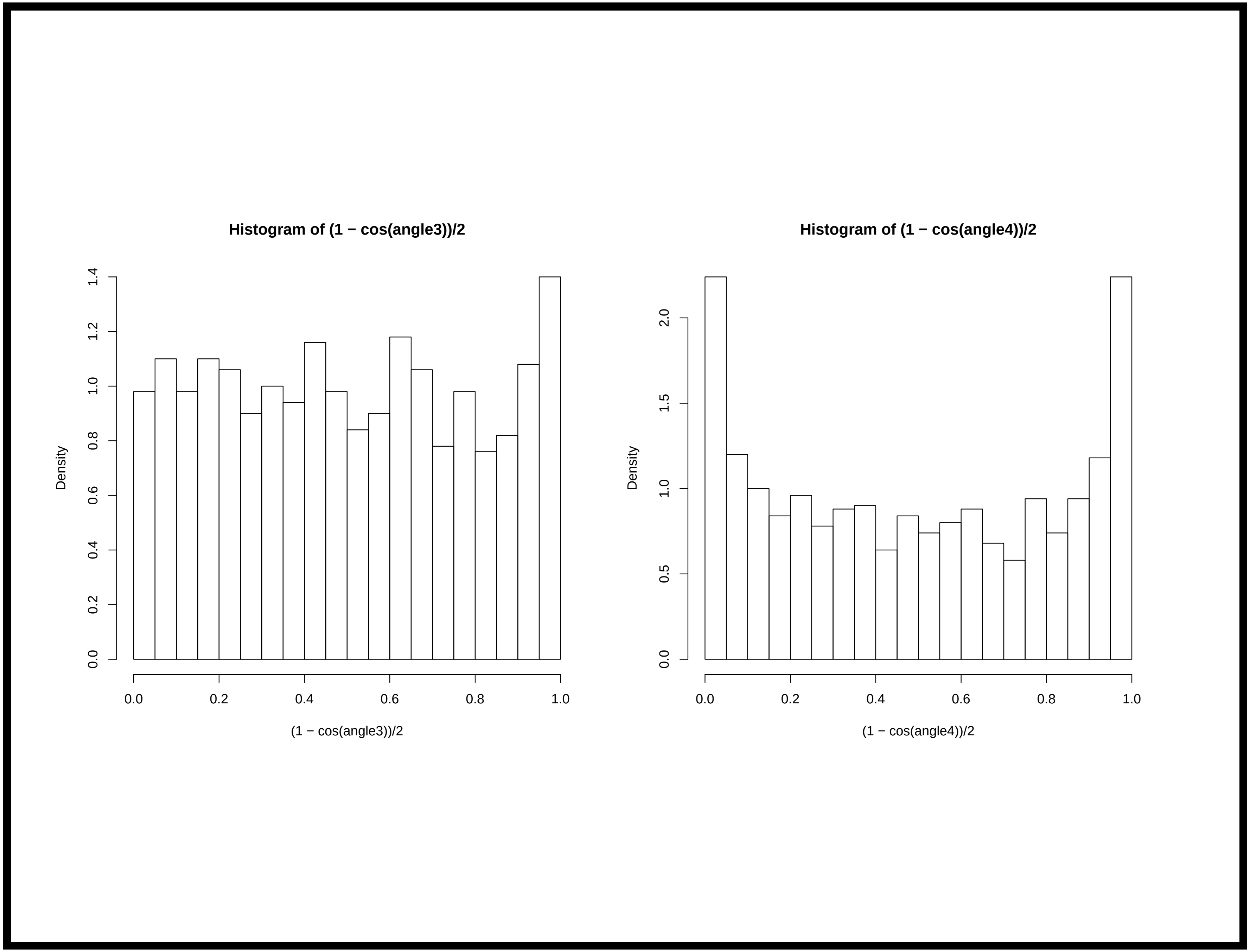
Simulations of SLE(κ) for $\kappa = 1, 2, 3, 4, 5, 6, 7, 7.9$ stopped when reaching the disk of radius 1

$n = 1000$ trials for each value of κ

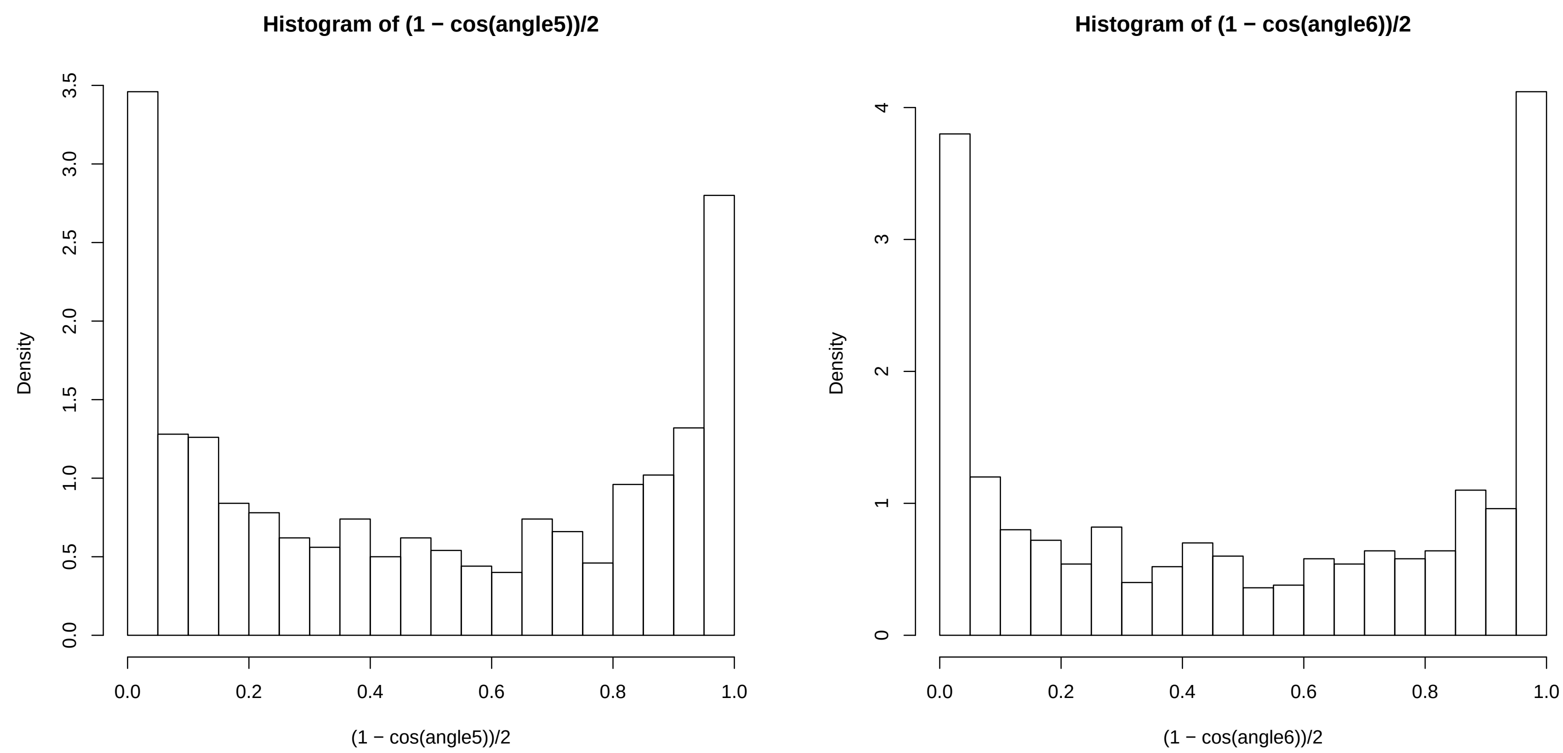
Measured angle $\Theta = \arg(\gamma(\tau))$ and plotted histogram of $X = \frac{1-\cos(\Theta)}{2} \in [0, 1]$

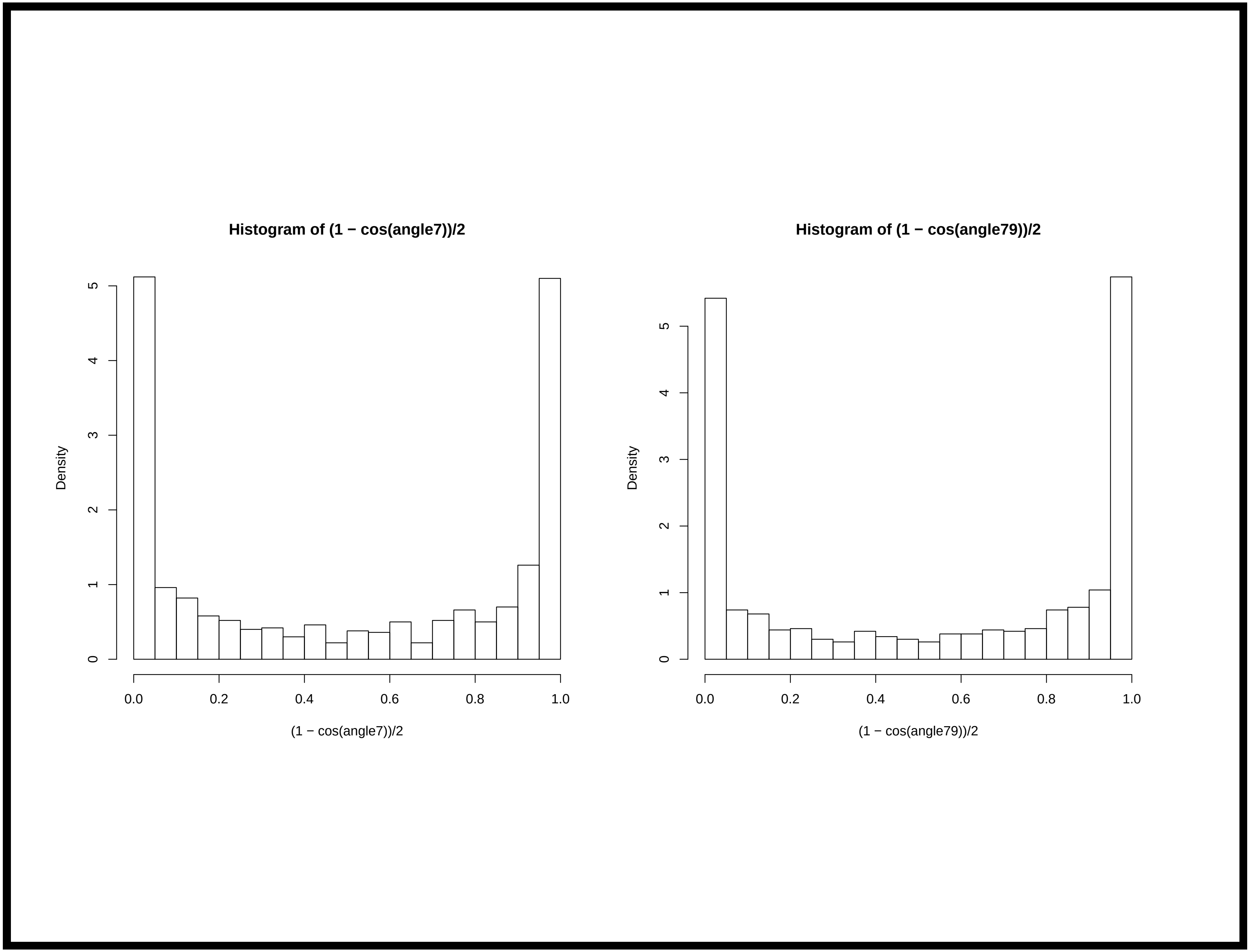




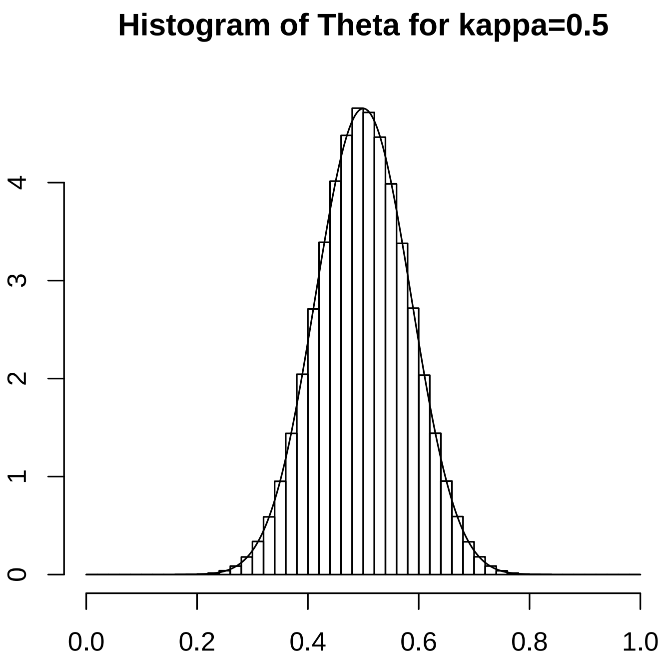


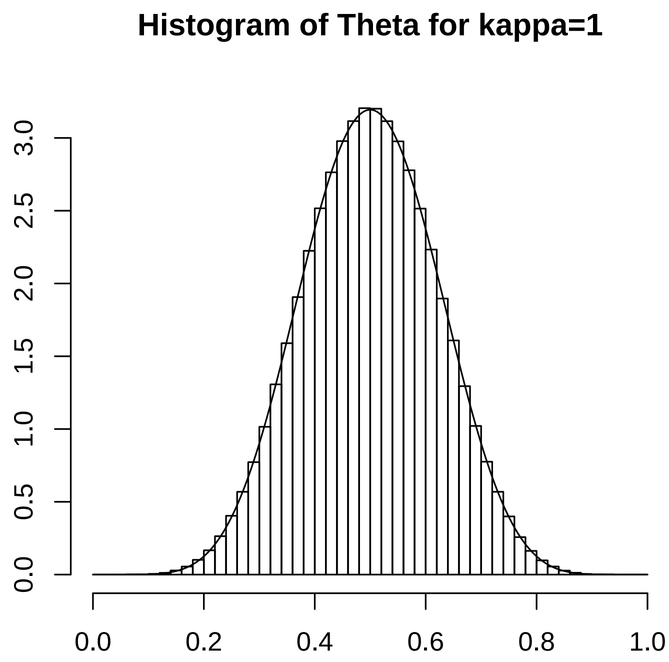
Histogram of (1 − cos(angle3))/2
Density
(1 − cos(angle3))/2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Histogram of (1 − cos(angle4))/2
Density
(1 − cos(angle4))/2
0.0
0.5
1.0
1.5
2.0

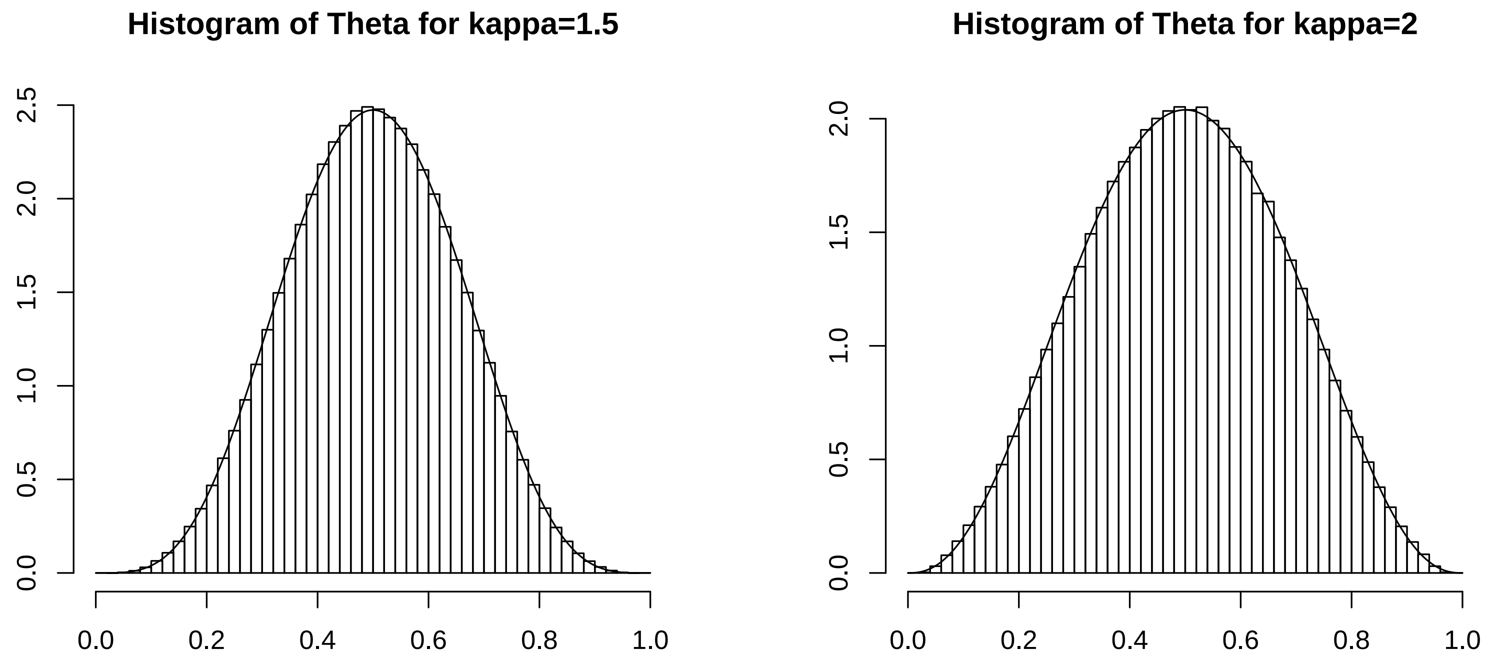

Histogram of (1 − cos(angle5))/2
Density
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0.0
0.2
0.4
0.6
0.8
1.0
(1 − cos(angle5))/2
Histogram of (1 − cos(angle6))/2
Density
0
1
2
3
4
0.0
0.2
0.4
0.6
0.8
1.0
(1 − cos(angle6))/2

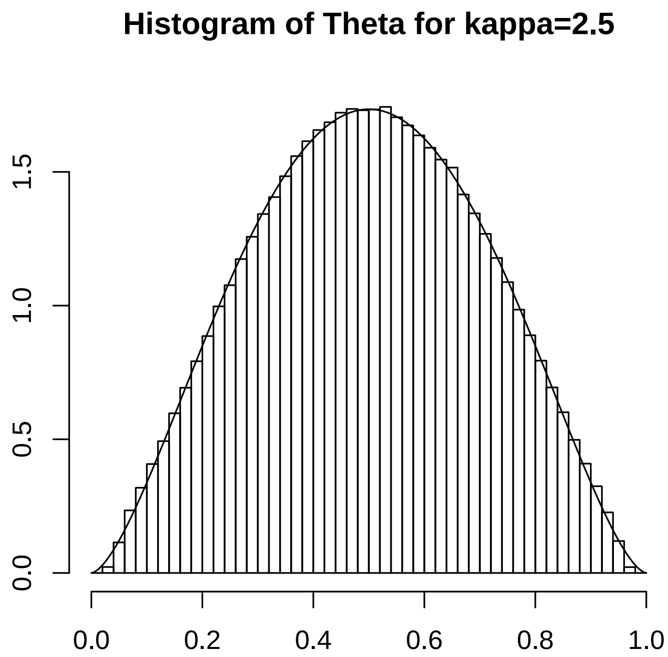

Histogram of (1 − cos(angle7))/2
Density
(1 − cos(angle7))/2
Histogram of (1 − cos(angle79))/2
Density
(1 − cos(angle79))/2

Simulations with $n = 1,000,000$ trials. These took time and are several years old. I have since changed my notation. Instead of theta in the title, it should be X.






Histogram of Theta for kappa=1.5
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
Histogram of Theta for kappa=2
0.0
0.5
1.0
1.5
2.0
0.0
0.2
0.4
0.6
0.8
1.0


Histogram of Theta for kappa=2.5
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0

Some Simulations

Simulations of SLE_κ for values of κ from 0.1 to 7.9 in increments of 0.1

$n = 1000$ trials for each value of κ

Measured angle $\Theta = \arg(\gamma(\tau))$ and considered random variable

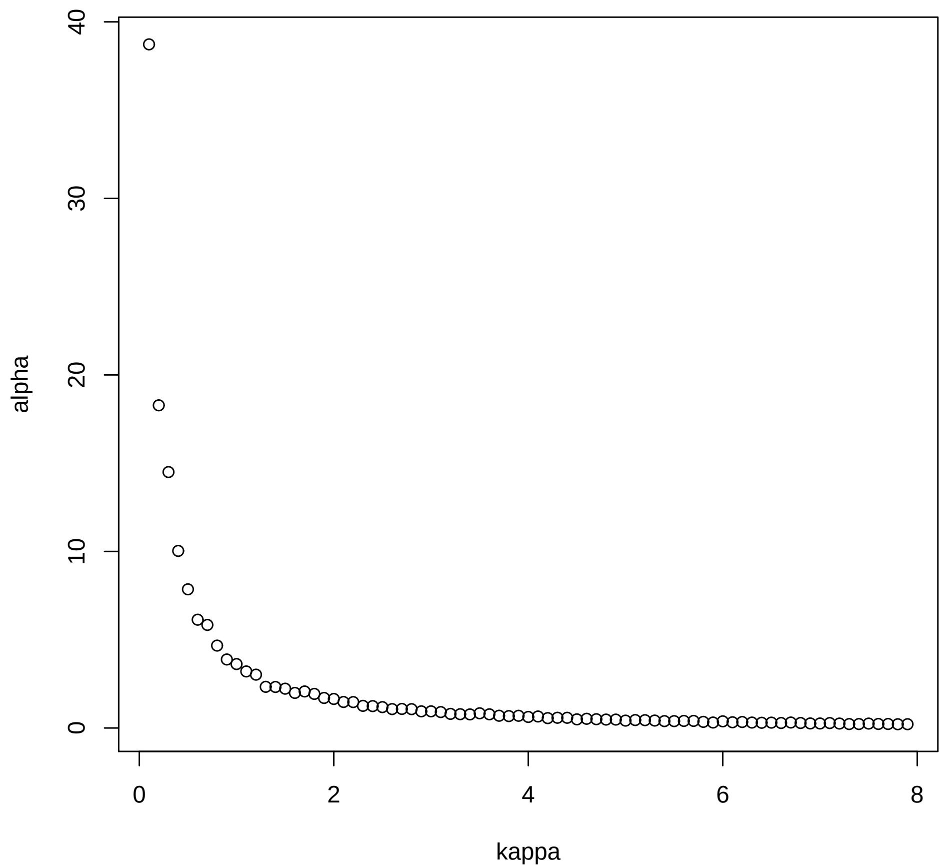
$$X = X_\kappa = \frac{1 - \cos(\Theta)}{2} \in [0, 1]$$

Assuming that $X_\kappa \sim \beta(\alpha(\kappa), \alpha(\kappa))$, estimated $\alpha(\kappa)$ by method-of-moments estimator

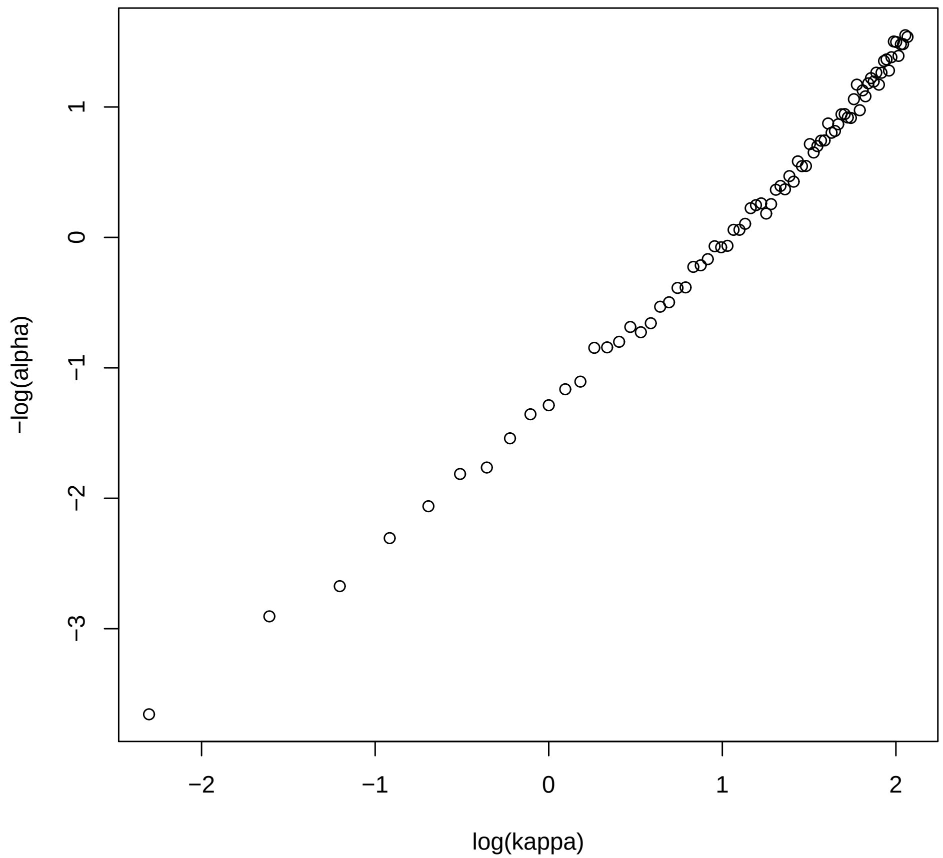
$$\hat{\alpha}(\kappa) = \overline{x}\left(\frac{\overline{x}(1-\overline{x})}{s^2} - 1\right)$$

where $\overline{x}$ is sample mean and s^2 is sample variance.

Plot of κ *vs.* $\hat{\alpha}(\kappa)$



Plot of $\log(\kappa)$ *vs.* $-\log(\hat{\alpha}(\kappa))$




```
> cor(log(kappa),-log(alpha))  0.9949292
```

What about working with Θ instead of X?

If the random variable Y has density function

$$f(y) = \frac{\sqrt{\pi}\,\Gamma(\alpha + 1/2)}{\Gamma(\alpha)} \sin^{2\alpha-1}(\pi y)$$

for $0 \le y \le 1$, then it is a standard exercise to show that

$$X = \frac{1 - \cos(\pi Y)}{2} \sim \beta(\alpha, \alpha).$$

I am comfortable working with beta random variables, but I am not sure how best to analyze the random variable $Y = \Theta/\pi$.