Estimates for the diameter of a chordal SLE path

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This talk is based on joint work with **Tom Alberts** of the Courant Institute:

Intersection probabilities for a chordal SLE path and a semicircle,
`arxiv:0707.3163`.

Some historical highlights

- **Stochastic Loewner Evolution** (SLE) introduced in 1999 by O. Schramm while considering possible scaling limits of 2-d LERW
- Developed by G. Lawler, O. Schramm, and W. Werner: proved dimension of boundary of Brownian island is $4/3$, proved LERW converges to SLE_2, proved UST Peano curve converges to SLE_8, limit of SAW must be $\text{SLE}_{8/3}$.
- S. Rohde and O. Schramm established basic properties of SLE, e.g., it is generated by a curve a.s.
- V. Beffara established Hausdorff dimension
- S. Smirnov proved Cardy's formula and that percolation exploration path converges to SLE_6 (for site percolation on triangular lattice)
- O. Schramm and S. Sheffield proved harmonic explorer converges to SLE_4; contour lines of discrete GFF converges to SLE_4
- Many other path properties under investigation: duality, reversibility, etc.
- Exciting links between SLE/CLE and Conformal Field Theory (CFT)/other stat mech models: percolation, $O(n)$-model/Ising model/Potts model, turbulence, spin glasses

Review of SLE

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane, and consider a simple (non-self-intersecting) curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$.

For every fixed $t \geq 0$, the slit plane $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0, t]$ is simply connected and so by the Riemann mapping theorem, there exists a conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$.

The map g_t is not unique, but we choose the unique one satisfying the hydrodynamic normalization $g_t(z) - z \to 0$ as $z \to \infty$.

It then follows that g_t can be expanded as

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty,$$

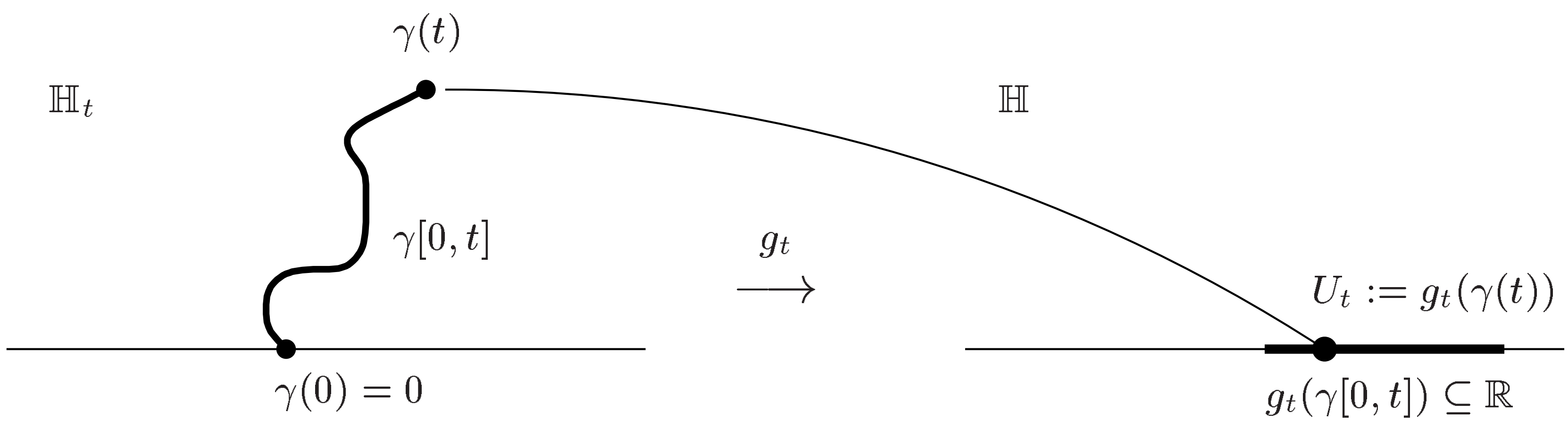
where $b(t) = \mathrm{hcap}(\gamma(0, t])$ is the **half-plane capacity** of γ up to time t.

The half-plane capacity is related to how likely a Brownian motion starting from infinity is to hit the curve before hitting the real line $\mathbb{R}$.

Review of SLE (cont)

For a slit plane such as $\mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t]$, the map g_t can be extended continuously to the boundary point $\gamma(t)$ of $\partial\mathbb{H}_t$.

With no additional assumptions on the simple curve γ, it can be shown that there is a unique point $U_t \in \mathbb{R}$ for all $t \geq 0$ with $U_t := g_t(\gamma(t))$ and that the function $t \mapsto U_t$ is continuous.



Review of SLE (cont)

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by a differential equation involving U_t.

This is due to C. Loewner who showed in 1923 that if γ is a curve as above such that its half-plane capacity $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$, then for $z \in \mathbb{H}$ with $z \notin \gamma[0, \infty)$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the partial differential equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{LE}$$

Note that if $b(t) \in C^1$ is an increasing function, then we can reparametrize the curve γ so that $\text{hcap}(\gamma(0, t]) = b(t)$. This is the so-called **parametrization by capacity**.

Review of SLE (cont)

The obvious thing to do now is to start with a continuous function $t \mapsto U_t$ from $[0,\infty)$ to $\mathbb{R}$ and solve the Loewner equation for g_t.

Ideally, we would like to solve (LE) for g_t, define simple curves $\gamma(t)$, $t \geq 0$, by setting $\gamma(t) = g_t^{-1}(U_t)$, and have g_t map $\mathbb{H} \setminus \gamma(0,t]$ conformally onto $\mathbb{H}$.

Although this is the correct intuition, it is not quite precise because we see from the denominator on the right-side of (LE) that problems can occur if $g_t(z) - U_t = 0$.

Formally, if we let T_z be the supremum of all t such that the solution to (LE) is well-defined up to time t with $g_t(z) \in \mathbb{H}$, and we define $\mathbb{H}_t = \{z : T_z > t\}$, then g_t is the unique conformal transformation of $\mathbb{H}_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $t \to \infty$.

Review of SLE (cont)

The novel idea of Schramm was to take the continuous function U_t to be a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

The **chordal Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization** (or simply SLE_κ) is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t}\, g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}\, W_t}, \quad g_0(z) = z, \tag{LE}$$

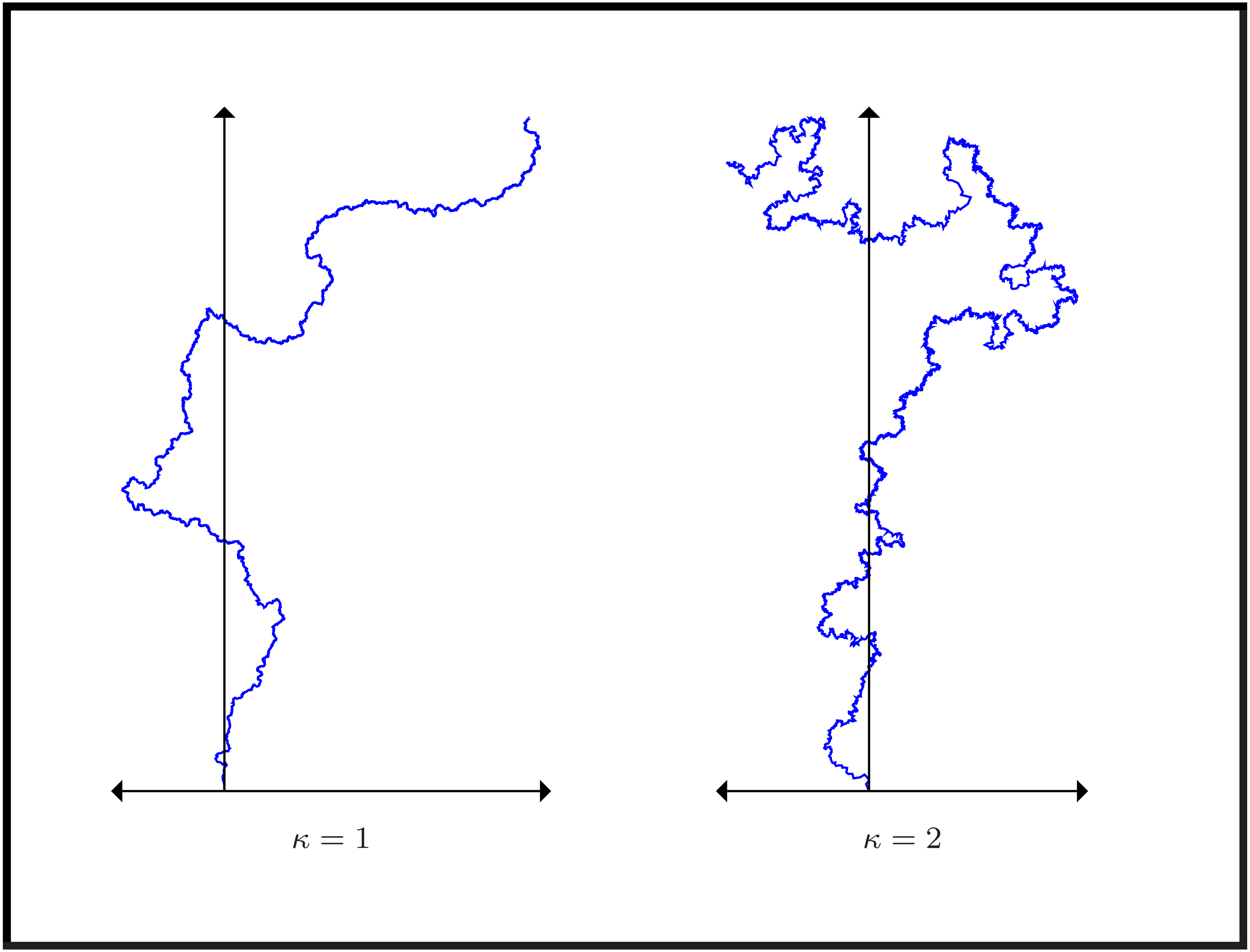
where W_t is a standard one-dimensional Brownian motion.

Review of SLE (cont)

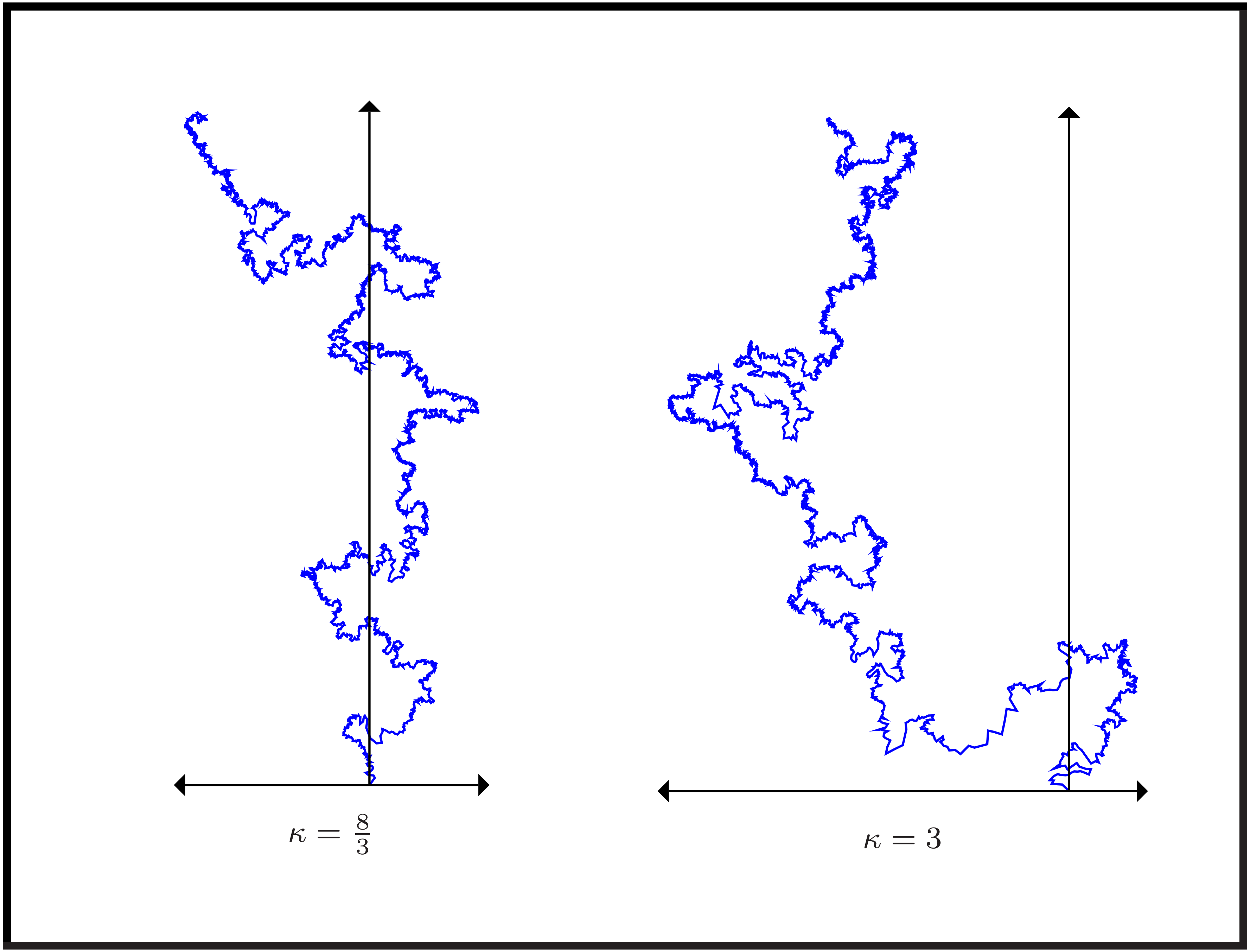
The question is now whether there exists a curve associated with the maps g_t.

- If $0 < \kappa \leq 4$, then there exists a random simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$, i.e., the curve $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$ never re-visits $\mathbb{R}$. As well, the maps g_t obtained by solving (LE) are conformal transformations of $\mathbb{H} \setminus \gamma(0, t]$ onto $\mathbb{H}$. For this range of κ, our intuition matches the theory!
- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$. These curves have double points and they do hit $\mathbb{R}$, but they never cross themselves! As such, $\mathbb{H} \setminus \gamma(0, t]$ is not simply connected. However, $\mathbb{H} \setminus \gamma(0, t]$ does have a unique connected component containing ∞. This is $\mathbb{H}_t$ and the maps g_t are conformal transformations of $\mathbb{H}_t$ onto $\mathbb{H}$. We think of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ where K_t is the hull of $\gamma(0, t]$ visualized by taking $\gamma(0, t]$ and filling in the holes.
- For $\kappa \geq 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ which is space-filling! Furthermore, it has double points, but does not cross itself! As in the case $4 < \kappa < 8$, the maps g_t are conformal transformations of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ where K_t is the hull of $\gamma(0, t]$.

As a result, we also refer to the curve γ as chordal SLE_κ. SLE paths are extremely rough: the Hausdorff dimension of a chordal SLE_κ path is $\min\{1 + \kappa/8, 2\}$.



$\kappa = 1$
$\kappa = 2$


$\kappa = \frac{8}{3}$
$\kappa = 3$

Review of SLE (cont)

Since there exists a curve γ associated with the maps g_t, it is possible to reparametrize it.

It can be shown that if U_t is a standard one-dimensional Brownian motion, then the solution to the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2/\kappa}{g_t(z) - U_t} = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

is chordal SLE_κ parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t/\kappa = at$.

Finally, chordal SLE as we have defined it can also be thought of as a measure on paths in the upper half plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

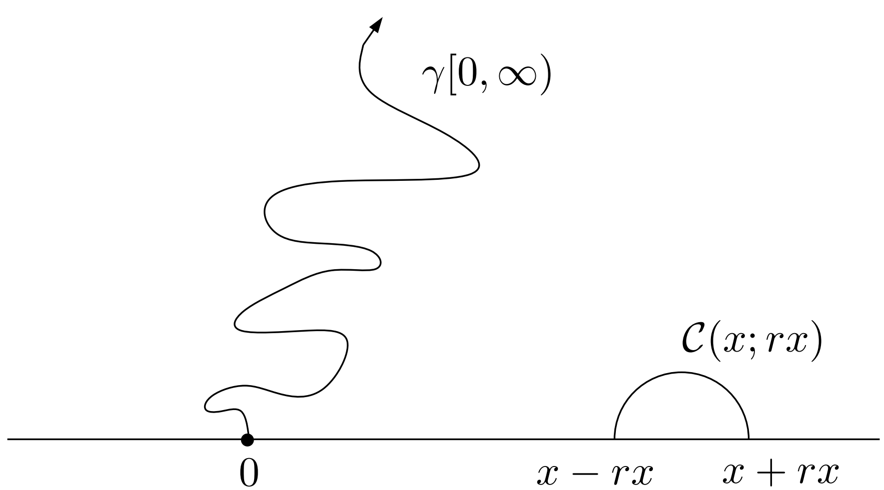
SLE is conformally invariant and so we can define chordal SLE_κ in any simply connected domain D connecting distinct boundary points z and w to be the image of chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation from $\mathbb{H}$ onto D sending $0 \mapsto z$ and $\infty \mapsto w$.

The main result

Theorem. Let $x > 0$ be real, $0 < r \le 1/3$, and $\mathcal{C}(x; rx) = \{x + rxe^{i\theta} : 0 < \theta < \pi\}$ denote the semicircle of radius rx centred at x in the upper half plane, and suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞.

(a) If $0 < \kappa < 8$, then $P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \asymp r^{\frac{8-\kappa}{\kappa}}$.

(b) If $\kappa = 8/3$, then $P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} = 1 - (1 - r^2)^{5/8} \sim \frac{5}{8} r^2$.

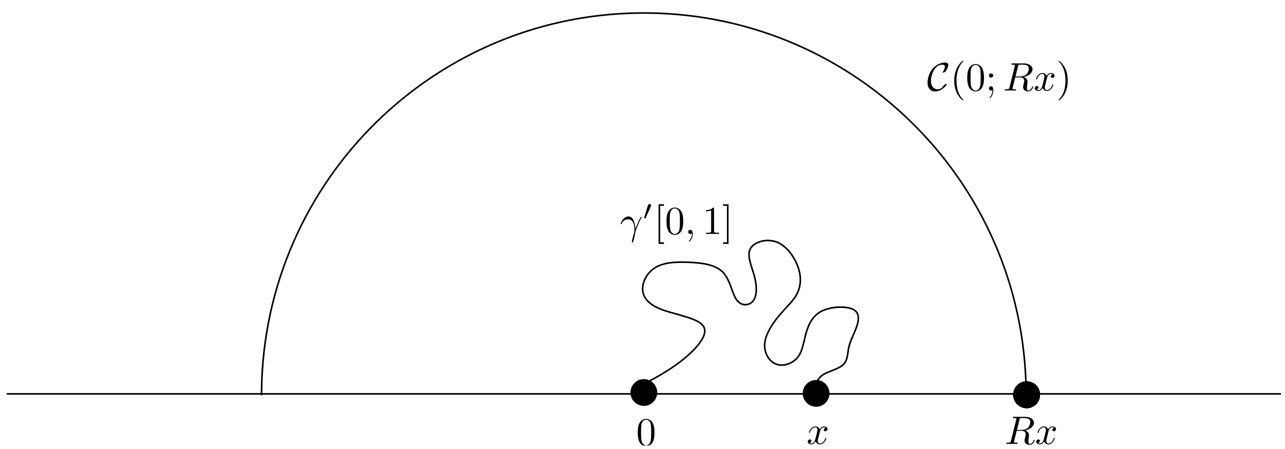


An equivalent formulation

Corollary. Let $x > 0$ be real, $R \geq 3$, and $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$ denote the circle of radius Rx centred at 0 in the upper half plane, and suppose that $\gamma' : [0,1] \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to x.

(a) If $0 < \kappa < 8$, then $P\{\gamma'[0,1] \cap \mathcal{C}(0; Rx) \neq \emptyset\} \asymp R^{\frac{\kappa-8}{\kappa}}$.

(b) If $\kappa = 8/3$, then $P\{\gamma'[0,1] \cap \mathcal{C}(0; Rx) \neq \emptyset\} = 1 - (1 - R^{-2})^{5/8} \sim \frac{5}{8} R^{-2}$.



Derivation of the corollary

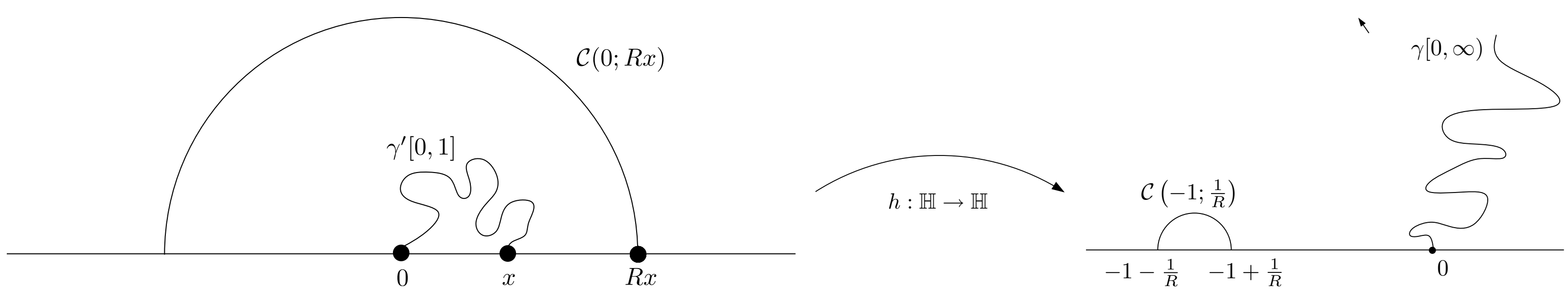
The idea is to determine the appropriate sequence of conformal transformations and use the conformal invariance of chordal SLE.

Suppose that $\gamma' : [0,1] \to \overline{\mathbb{H}}$ is an SLE$_\kappa$ in $\mathbb{H}$ from 0 to $x > 0$. Note that we are not interested in the parametrization of the SLE path, but only in the points visited by its trace. Suppose that $R \geq 3$, and consider $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$. For $z \in \mathbb{H}$, let

$$h(z) = \frac{R^2}{R^2 - 1} \frac{z}{x - z}$$

so that $h : \mathbb{H} \to \mathbb{H}$ is a conformal (Möbius) transformation with $h(0) = 0$ and $h(x) = \infty$. It is straightforward (though tedious) to verify that

$$h\left(\mathcal{C}(0; Rx)\right) = \mathcal{C}\left(-1; \frac{1}{R}\right).$$



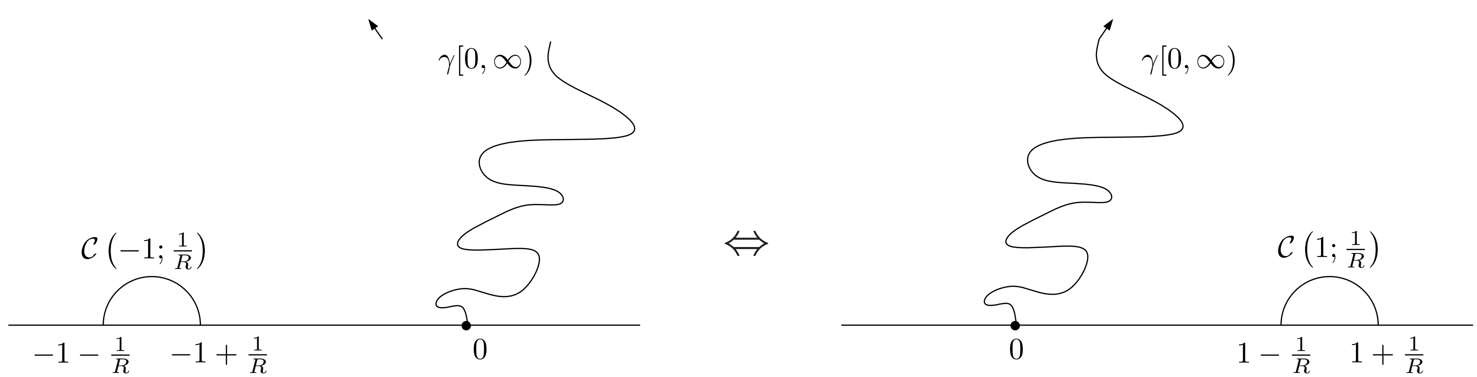
Derivation of the corollary (cont.)

If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then the conformal invariance of SLE implies that

$$P\{\gamma'[0,1] \cap \mathcal{C}(0;Rx) \neq \emptyset\} = P\{h(\gamma'[0,1]) \cap h(\mathcal{C}(0;Rx)) \neq \emptyset\}$$
$$= P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(-1, \frac{1}{R}\right) \neq \emptyset\right\}.$$

By the symmetry of SLE about the imaginary axis,

$$P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(-1, \frac{1}{R}\right) \neq \emptyset\right\} = P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(1, \frac{1}{R}\right) \neq \emptyset\right\} \asymp R^{1-4a}.$$



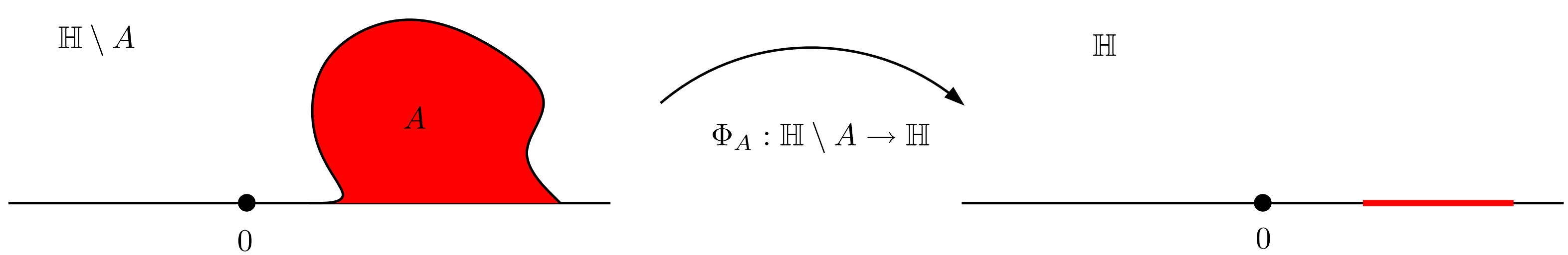
The $\kappa = 8/3$ case

The key fact that is needed is the restriction property of chordal $\text{SLE}_{8/3}$.

Proposition. [Lawler-Schramm-Werner] If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal $\text{SLE}_{8/3}$ in $\mathbb{H}$ from 0 to ∞, and A is a bounded subset of $\mathbb{H}$ such that $\mathbb{H} \setminus A$ is simply connected, $A = \mathbb{H} \cap \overline{A}$, and $0 \notin \overline{A}$, then

$$P\{\gamma[0, \infty) \cap A = \emptyset\} = \left[\Phi'_A(0)\right]^{5/8}$$

where $\Phi_A : \mathbb{H} \setminus A \to \mathbb{H}$ is the unique conformal transformation of $\mathbb{H} \setminus A$ to $\mathbb{H}$ with $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$.

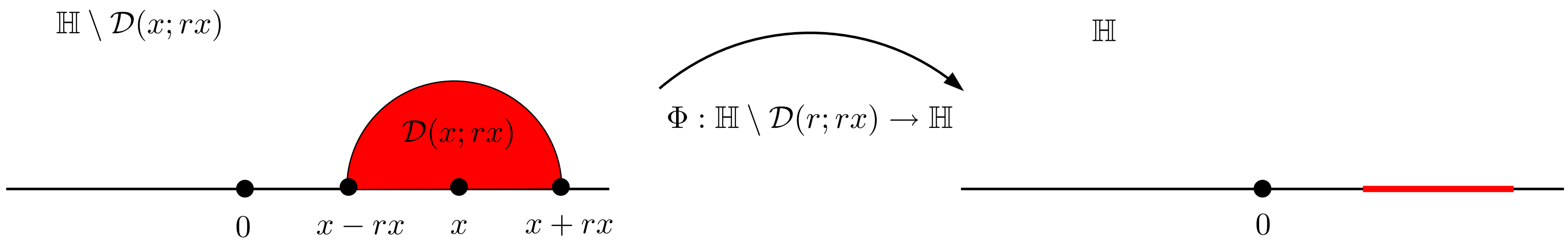


The $\kappa = 8/3$ case (cont.)

This implies that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;rx) = \emptyset\} = \left[\Phi'(0)\right]^{5/8}$$

where $\Phi = \Phi_{\mathcal{D}(x;rx)}(z)$ is the conformal transformation from $\mathbb{H} \setminus \mathcal{D}(x;rx)$ onto $\mathbb{H}$ with $\Phi(0) = 0$ and $\Phi(z) \sim z$ as $z \to \infty$.



In fact, the exact form of $\Phi(z)$ is given by

$$\Phi(z) = z + \frac{r^2x^2}{z - x} + r^2x.$$

Note that $\Phi(0) = 0$, $\Phi(\infty) = \infty$, and $\Phi'(\infty) = 1$. We calculate $\Phi'(0) = 1 - r^2$ and therefore conclude that

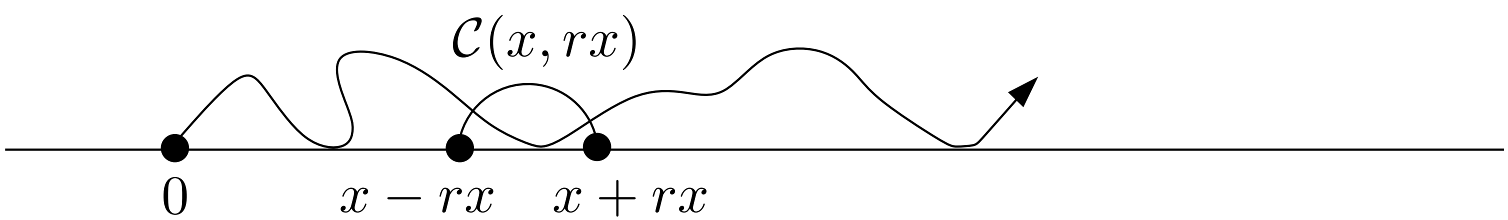
$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;rx) = \emptyset\} = (1 - r^2)^{5/8}.$$

The lower bound for $4 < \kappa < 8$

Theorem. Let $0 < r \leq 1/3$ and $x > 0$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$, then there exists a constant c_a such that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \geq c_a r^{4a-1}.$$

Proof. It is clear that if $\gamma[0, \infty)$ intersects the interval $[x - rx, x + rx]$ then it also intersects the semicircle $\mathcal{C}(x; rx)$.



The lower bound for $4 < \kappa < 8$ (cont.)

By a result of Rohde and Schramm (a generalized Cardy's formula) and the scale invariance of SLE,

$$
\begin{aligned}
&P\left\{\gamma[0,\infty) \cap [x - rx, x + rx] \neq \emptyset\right\} \\
&\qquad = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{\frac{2r}{1+r}} \frac{dt}{t^{2-4a}(1-t)^{2a}} \\
&\qquad \geq \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{\frac{2r}{1+r}} \frac{dt}{t^{2-4a}(1/2)^{2a}} \\
&\qquad \geq \frac{\Gamma(2a)2^{2a}}{\Gamma(1-4a)\Gamma(4a)} (2r)^{4a-1}.
\end{aligned}
$$

The first and second inequalities use $0 < r \leq 1/3$.

Note: $4 < \kappa < 8$ iff $0 < 2 - 4a < 1$

The lower bound for $0 < \kappa \le 4$

Theorem. Let $0 < r < 1$ and $x > 0$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa \le 4$ and $a = 2/\kappa$, then there exists a constant c_a such that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \ge c_a r^{4a-1}.$$

To prove the result, we recall the probability that a fixed point $z \in \mathbb{H}$ lies to the left of $\gamma[0, \infty)$. The version that we include may be found in Garban and Trujillo Ferreras and is equivalent to the one given by Schramm.

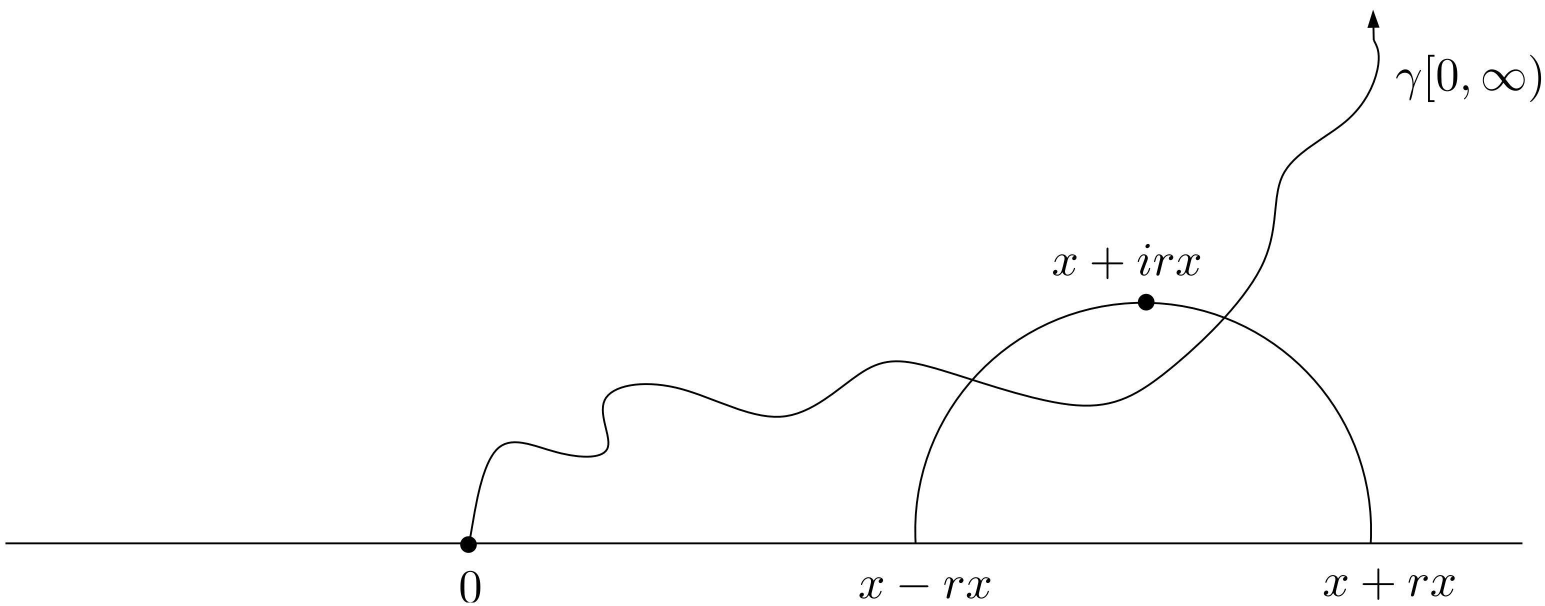
Proposition. [Schramm] Let $z = \rho e^{i\theta} \in \mathbb{H}$ and $f(z) = P\{z \text{ is to the left of } \gamma[0,\infty)\}$. By scaling, the function f only depends on θ and is given by

$$f(\theta) = \frac{\int_0^\theta (\sin\alpha)^{4a-2}\, d\alpha}{\int_0^\pi (\sin\alpha)^{4a-2}\, d\alpha}.$$

The lower bound for $0 < \kappa \leq 4$ (cont.)

Proof. The figure clearly shows that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \geq P\{x + irx \text{ is to the left of } \gamma[0,\infty)\}.$$



The lower bound for $0 < \kappa \leq 4$ (cont.)

Since $\arg(x + irx) = \arctan(r)$ and since $2 \sin t \geq t$ for $0 \leq t \leq \pi/4$, we conclude from the Proposition that

$$\begin{aligned} P\{x + irx \text{ is to the left of } \gamma[0,\infty)\} \cdot \int_0^\pi (\sin\alpha)^{4a-2}\, d\alpha \\ = \int_0^{\arctan(r)} (\sin\alpha)^{4a-2}\, d\alpha \\ \geq \frac{1}{2} \int_0^{\arctan(r)} \alpha^{4a-2}\, d\alpha \\ = \frac{\arctan^{4a-1}(r)}{8a-2}. \qquad (*) \end{aligned}$$

Since $8 \arctan t \geq \pi t$ for $0 \leq t \leq 1$, we see that $(*)$ implies that there exists a constant c_a, namely

$$c_a = \frac{\pi^{4a-1}}{4^{6a-1}(4a-1) \int_0^\pi (\sin\alpha)^{4a-2}\, d\alpha},$$

such that $P\{x + irx \text{ is to the left of } \gamma[0,\infty)\} \geq c_a r^{4a-1}$.

The upper bound

Theorem. Let $0 < r \leq 1/3$ and $x > 0$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$, then there exists a constant c_a such that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \leq c_a r^{4a-1}.$$

The tool we need to establish the upper bound is due to Beffara.

Proposition. [Beffara] If $z \in \mathbb{H}$, $0 < \epsilon \leq \Im\{z\}/2$, and $\mathcal{B}(z;\epsilon) = \{w \in \mathbb{C} : |z - w| < \epsilon\}$ denotes the ball of radius ϵ centred at z, then

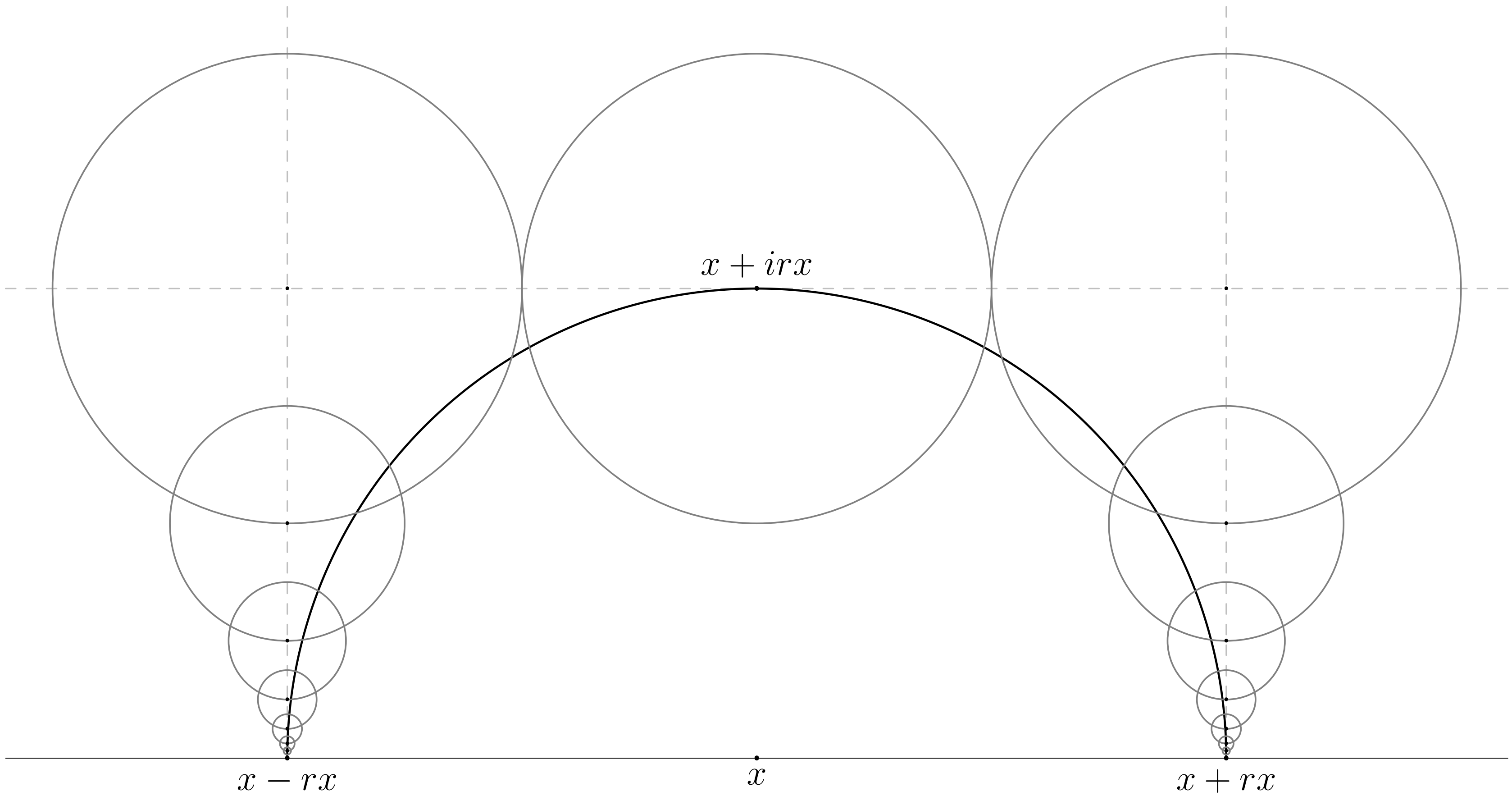
$$P\{\gamma[0,\infty) \cap \mathcal{B}(z;\epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{\Im\{z\}}\right)^{1-\frac{1}{4a}} \left(\frac{\Im\{z\}}{|z|}\right)^{4a-1}$$

where the constants implied by $\asymp$ may depend on a.

Remark. We would like to stress that the Proposition holds for all $a > 1/4$ (equivalently, all $0 < \kappa < 8$).

The upper bound (cont.)

Our general strategy will be to cover the semicircle $\mathcal{C}(x;rx)$ with a sequence of balls and then apply the Proposition to each ball.



The upper bound (cont.)

Proof. Set $z_0 = x + irx$ and for $n = 1, 2, \ldots$, let $z_{\pm n} = x \pm rx + irx2^{-|n|+1}$. Using the Proposition,

$$P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_{\pm n}; \frac{\Im\{z_{\pm n}\}}{2}\right) \neq \emptyset\right\} \asymp 2^{\frac{1}{4a}-1}\left(\frac{\Im\{z_{\pm n}\}}{|z_{\pm n}|}\right)^{4a-1} \asymp \frac{r^{4a-1}}{2^{(4a-1)|n|}}$$

since $|z_{\pm n}| \asymp x$ for $0 < r \leq 1/3$. Hence,

$$\sum_{n=-\infty}^{\infty} P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_n; \frac{\Im\{z_n\}}{2}\right) \neq \emptyset\right\} \asymp r^{4a-1}. \qquad (*)$$

But if $\gamma[0,\infty)$ intersects $\mathcal{C}(x; rx)$, then it also must intersect (at least) one of $\mathcal{B}\left(z_{\pm n}; \frac{\Im\{z_{\pm n}\}}{2}\right)$, as is clear from the Figure. Hence, $(*)$ implies that there exists a constant c_a such that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \leq \sum_{n=-\infty}^{\infty} P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_n; \frac{\Im\{z_n\}}{2}\right) \neq \emptyset\right\} \leq c_a r^{4a-1}.$$

Rephrasing the main result

Theorem. Let $x > 0$ be a fixed real number, and suppose $0 < \epsilon \le x/3$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$, then

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;\epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}$$

where $\mathcal{C}(x;\epsilon)$ is the semicircle of radius ϵ centred at x in the upper half plane.

Written in this form, it is seen to generalize the result of Rohde and Schramm who prove that for $4 < \kappa < 8$,

$$P\{\gamma[0,\infty) \cap [x-\epsilon, x+\epsilon] \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}.$$

An application

Let $0 < r \le 1/3$, and suppose that $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$.

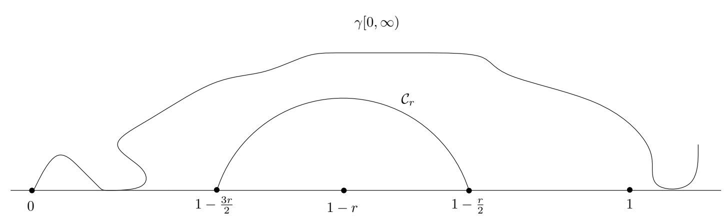
Theorem. There exist constants c'_a and c''_a such that

$$1 - c'_a r^{4a-1} \le \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\} \le \sup_{z \in \mathcal{C}_r} P\{T_z = T_1\} \le 1 - c''_a r^{4a-1}$$

where

$$\mathcal{C}_r = \mathcal{C}\left(1 - r; \frac{r}{2}\right)$$

denotes the circle of radius $r/2$ centred at $1 - r$ in the upper half plane.



Corollary. There exist constants c'_a and c''_a such that

$$1 - c'_a r^{4a-1} \le P\{T_z = T_1 \text{ for all } z \in \mathcal{C}_r\} \le 1 - c''_a r^{4a-1}.$$

Proof of the application

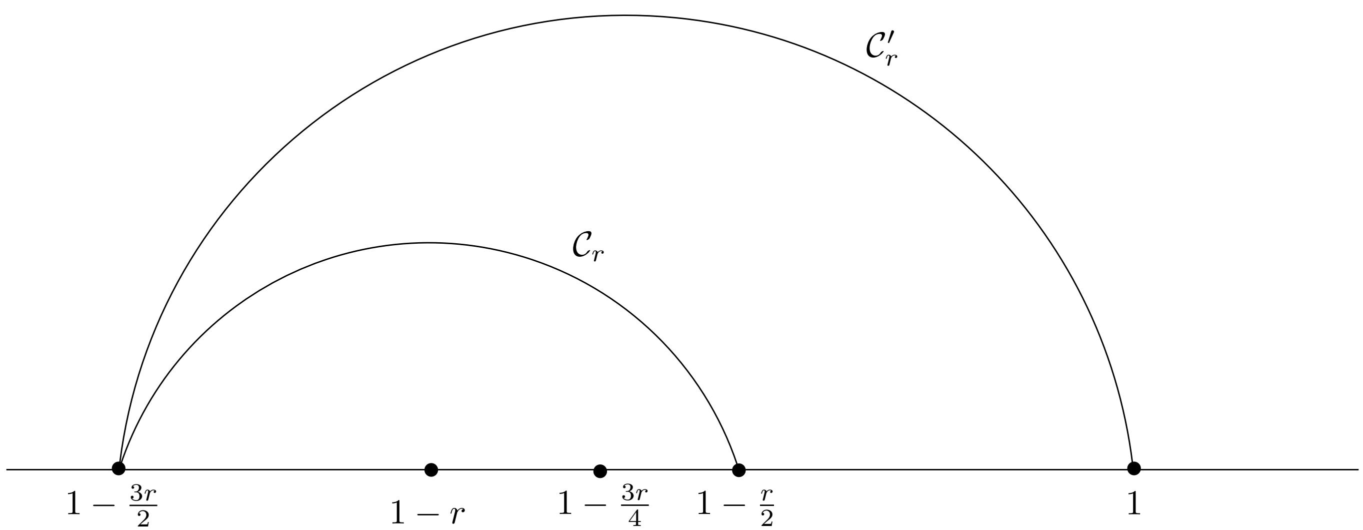
The proof follows by combining the main result with a method due to Dubédat.

Suppose that $0 < r \leq 1/3$ and consider the two semicircles

$$\mathcal{C}_r = \mathcal{C}\left(1 - r; \frac{r}{2}\right)$$

and

$$\mathcal{C}'_r = \mathcal{C}\left(1 - \frac{3r}{4}; \frac{3r}{4}\right).$$



Proof of the application (lower bound)

It follows from the rephrased main result that

$$P\{\gamma[0,\infty)\cap\mathcal{C}_r'\neq\emptyset\}\asymp r^{4a-1}$$

and so there exists a constant c_a' such that

$$1-c_a'r^{4a-1}\leq P\{\gamma[0,\infty)\cap\mathcal{C}_r'=\emptyset\}.$$

However, it clearly follows that

$$P\{\gamma[0,\infty)\cap\mathcal{C}_r'=\emptyset\}\leq\inf_{z\in\mathcal{C}_r}P\{T_z=T_1\}$$

where T_z is the swallowing time of the point $z\in\overline{\mathbb{H}}$ (and the infimum is over all $z\in\mathcal{C}_r$ *not* $z\in\mathcal{C}_r'$). From this we conclude that there exists a constant c_a' such that

$$1-c_a'r^{4a-1}\leq\inf_{z\in\mathcal{C}_r}P\{T_z=T_1\}.$$

Proof of the application (upper bound)

In order to derive an upper bound, we use a method due to Dubédat.

Let g_t denote the solution to the chordal Loewner equation with driving function $U_t = -B_t$ where B_t is a standard one-dimensional Brownian motion with $B_0 = 0$. For $t < T_1$, the swallowing time of the point 1, consider the conformal transformation $\tilde{g}_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ given by

$$\tilde{g}_t(z) = \frac{g_t(z) + B_t}{g_t(1) + B_t}, \quad \tilde{g}_0(z) = z.$$

Note that $\tilde{g}_t(\gamma(t)) = 0$, $\tilde{g}_t(1) = 1$, $\tilde{g}_t(\infty) = \infty$, and that $\tilde{g}_t(z)$ satisfies the stochastic differential equation

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] \frac{dt}{(g_t(1) + B_t)^2} + [1 - \tilde{g}_t(z)] \frac{dB_t}{g_t(1) + B_t}.$$

If we now perform a time-change and also denoted the time-changed flow by $\{\tilde{g}_t(z),\ t \geq 0\}$, then then $\tilde{g}_t(z)$ satisfies the SDE

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] dt + [1 - \tilde{g}_t(z)]\, dB_t$$

Dubédat shows that for all $\kappa > 0$, this does not explode in finite time (wp1).

Proof of the application (upper bound) (cont.)

Therefore, if F is an analytic function on $\mathbb{H}$ such that $\{F(\tilde{g}_t(z)),\ t \geq 0\}$ is a local martingale, then Itô's formula (at $t = 0$) implies that F must be a solution to the differential equation

$$w(1-w)F''(w) + [2a - (2-2a)w]F'(w) = 0.$$

An explicit solution is given by

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2} d\zeta$$

which is normalized so that $F(0) = 0$ and $F(1) = 1$.

Note that this is a Schwarz-Christoffel transformation of the upper half plane onto the isosceles triangle whose interior angles are $(1-2a)\pi$, $(1-2a)\pi$, and $(4a-1)\pi$.

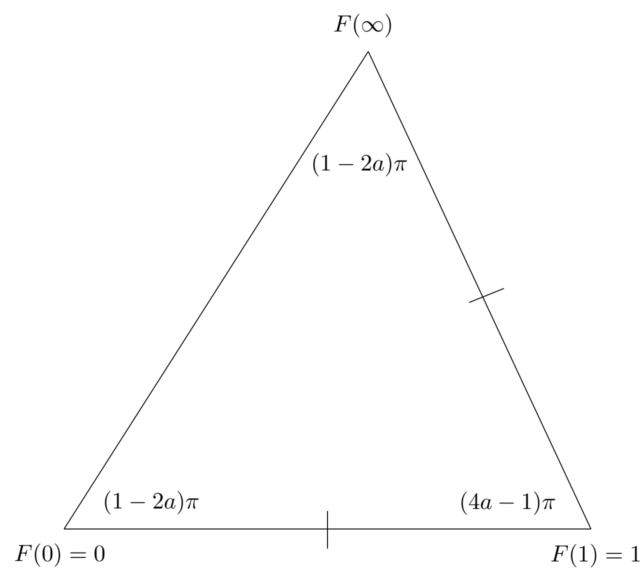
Proof of the application (upper bound) (cont.)

If

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2} d\zeta,$$

then the vertices of the traingle are at $F(0) = 0$, $F(1) = 1$, and

$$F(\infty) = \frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(2-4a)\Gamma(4a-1)} e^{(1-2a)\pi i}.$$



Proof of the application (upper bound) (cont.)

Apply the optional sampling theorem to the martingale $F(\tilde{g}_{t\wedge T_z \wedge T_1}(z))$ to find that for $z \in \mathbb{H}$,

$$\begin{aligned} F(\tilde{g}_0(z)) = F(z) &= F(0)P\{T_z < T_1\} + F(1)P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\} \\ &= P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}. \qquad (*) \end{aligned}$$

Consequently, identifying the imaginary and real parts of $(*)$ implies that

$$\Re\{F(z)\} = P\{T_z = T_1\} + \Re\{F(\infty)\}P\{T_z > T_1\}.$$

Since $\Re\{F(\infty)\} \geq 0$, we conclude $P\{T_z = T_1\} \leq \Re\{F(z)\} \leq |F(z)|$.

But now integrating along the straight line from 0 to z gives

$$|F(z)| \leq 1 - \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_{|z|}^{1} \rho^{-2a}(1-\rho)^{4a-2} d\rho$$

which relied on the fact that $4a - 2 < 0$.

Proof of the application (upper bound) (cont.)

If $z \in \mathcal{C}_r$ so that $0 < 1 - \frac{3r}{2} \leq |z| \leq 1 - \frac{r}{2} < 1$ by definition, then

$$\int_{|z|}^{1} \rho^{-2a}(1-\rho)^{4a-2}\,d\rho \geq \frac{2^{1-4a}}{4a-1} r^{4a-1}.$$

Hence,

$$P\{T_z = T_1\} \leq |F(z)| \leq 1 - c_a'' r^{4a-1}$$

where

$$c_a'' = \frac{2^{1-4a}}{4a-1} \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)}.$$

Taking the supremum of the previous expression over all $z \in \mathcal{C}_r$ gives us the required upper bound.

Fin.