Convergence of 2D critical percolation to SLE_6

Michael J. Kozdron
University of Regina

`http://stat.math.uregina.ca/∼kozdron/`

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Abstract

In this lecture we present the main ideas of the convergence, in the scaling limit, of the critical site percolation exploration path on the triangular lattice to SLE_6. This example is one of only a select few where convergence to SLE of an appropriate discrete statistical mechanics model is completely understood. As such, the result is crucial for the determination of the critical exponents in two dimensions and in all applications of SLE to percolation. Our primary reference for this result is the recent paper of Camia and Newman [6].

Disclaimer

Until I started preparing this talk 6 weeks ago, I knew **very little** about percolation. I thought that attending the workshop would be a good way to learn about it!

The proof of the convergence of the percolation path to SLE_6 is quite technical, and our primary reference is the recent paper by Camia and Newman [6].

I will not be going through the technical elements of the proof. I don't know them all myself, and besides it would take too long if I did!

However, I will try to highlight the key ideas. My goal when I started reading the paper was to understand the statement of the theorem. My goal for this talk is to have you understand the statement of the theorem.

For anyone who is interested in the technical details, I leave it to you to consult the original paper.

W. Werner's notes from Park City

Note that a recent preprint by W. Werner [15] contains lecture notes from a short course given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics.

Lecture 3 in those notes is concerned with a proof of this convergence result, but Werner follows a different approach than Camia and Newman.

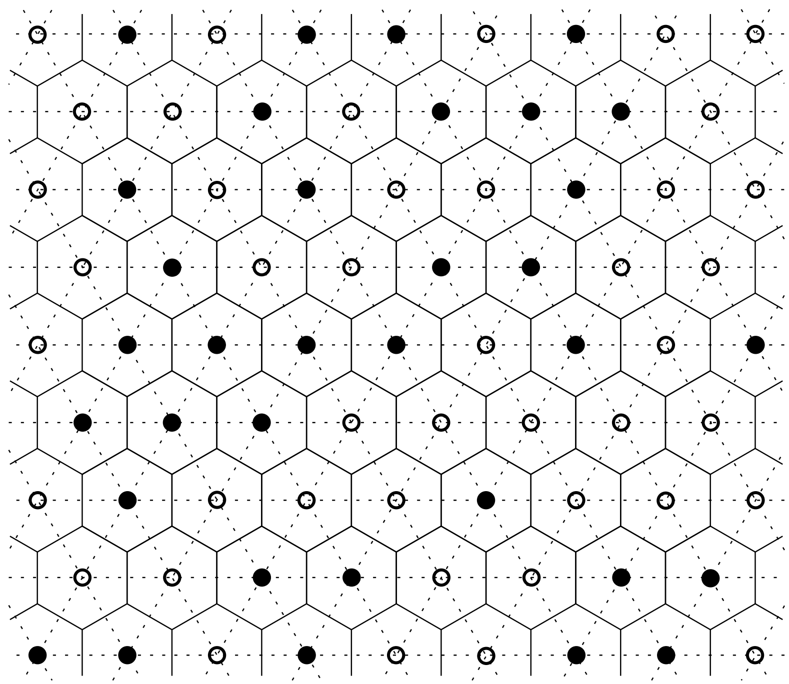
In fact, Werner's notes [15] contain six lectures and a set of exercises on critical site percolation on the triangular lattice that coincide with the topic of this Arbeitsgemeinschaft.

Triangular and hexagonal lattices

$\mathcal{T}$: standard two-dimensional triangular lattice with lattice spacing 1

$\mathcal{H}$: hexagonal lattice which is dual to $\mathcal{T}$ ($\delta\mathcal{H}$ indicates lattice spacing δ)

Note that site percolation on $\mathcal{T}$ corresponds to face percolation on $\mathcal{H}$.



Approximating domains

$D \subset \mathbb{C}$: bounded, simply connected Jordan domain (i.e., ∂D is a simple closed curve which is homeomorphic to the unit circle)

$D^\delta \subset \delta\mathcal{H}$: Jordan set which approximates D. i.e., D^δ is a simply connected subset of $\delta\mathcal{H}$ whose external site boundary is a simple closed loop of hexagons such that D^δ is a discrete approximation to D.

Let a, $b \in \partial D$ be distinct boundary points.

Let a^δ, $b^\delta \in \partial D^\delta$ be the corresponding external boundary vertices (or e-vertices).

Without being more precise about this exact approximation, we denote by (D, a, b) the simply connected Jordan domain with two distinguished boundary points, and let its δ-scale approximation be denoted by $(D^\delta, a^\delta, b^\delta)$.

Essentially, we think of choosing $D^\delta = D \cap \delta\mathcal{H}$. (But this may not produce a simply connected D^δ so we do need to be careful.)

Assume that D^δ, a^δ, and b^δ are chosen so that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ in the Carathéodory sense as $\delta \to 0$.

Defining the (critical site) percolation exploration path

Consider D^δ with distinguished e-vertices a^δ and b^δ.

These two distinguished boundary points partition the (topological) boundary of D^δ into two disjoint arcs.

Associate to all external boundary hexagons on one of the arcs the colour "red" and associate to all boundary hexagons on the other arc the colour "white."

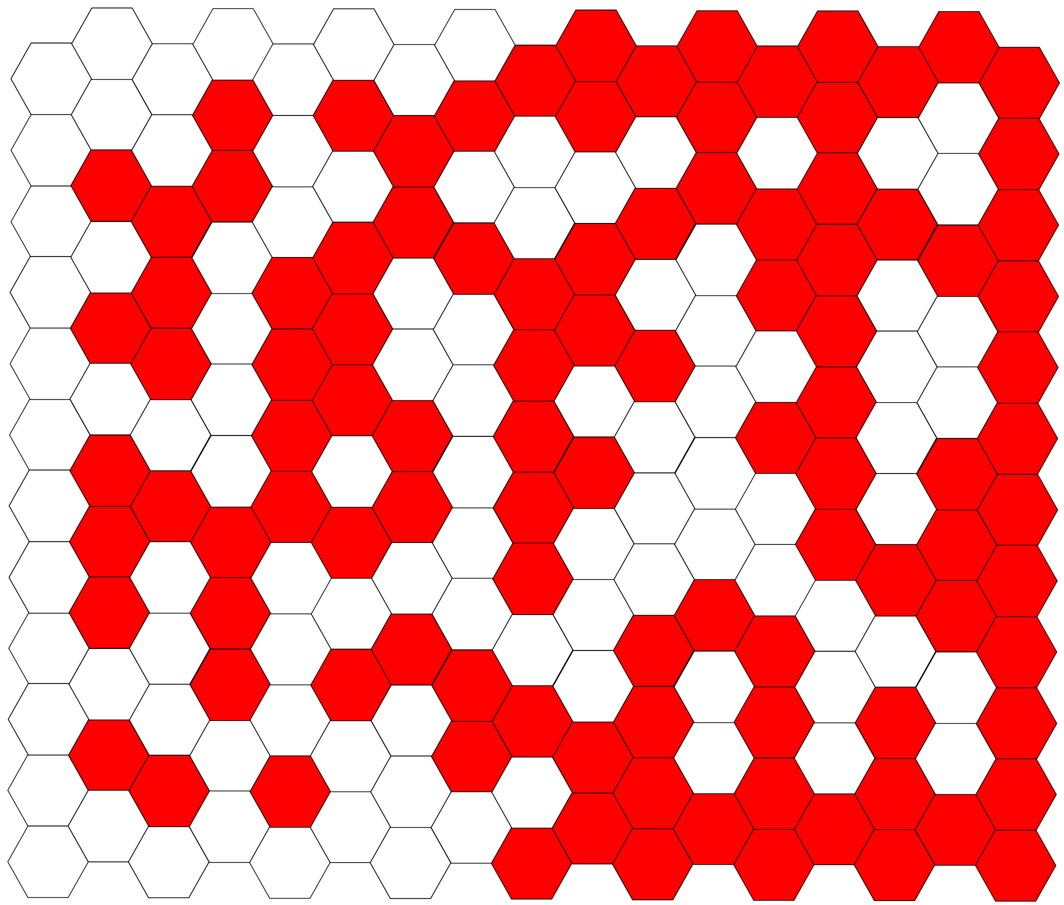
Perform critical site percolation on D^δ. That is, for each remaining interior hexagon colour it either red with probability $1/2$ or white with probability $1/2$.

There will be a resulting **interface** joining a^δ with b^δ; that is, a simple path connecting a^δ to b^δ with the property that all hexagons on one side of the path will be white while all hexagons on the other side of the path will be red.

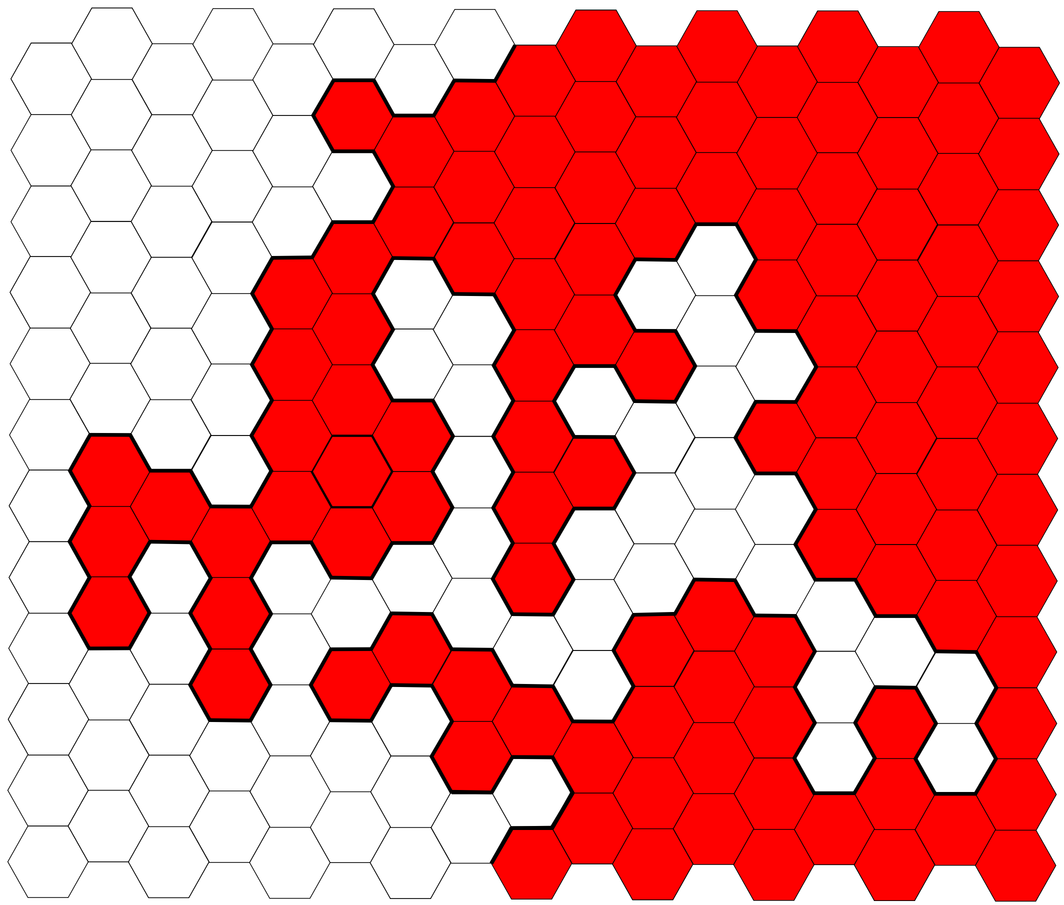
We call this path/interface the (critical site) percolation exploration path and denote it by $\gamma^\delta_{D,a,b}$.

As $\delta \downarrow 0$, it is this path that converges to chordal SLE_6 in D from a to b.

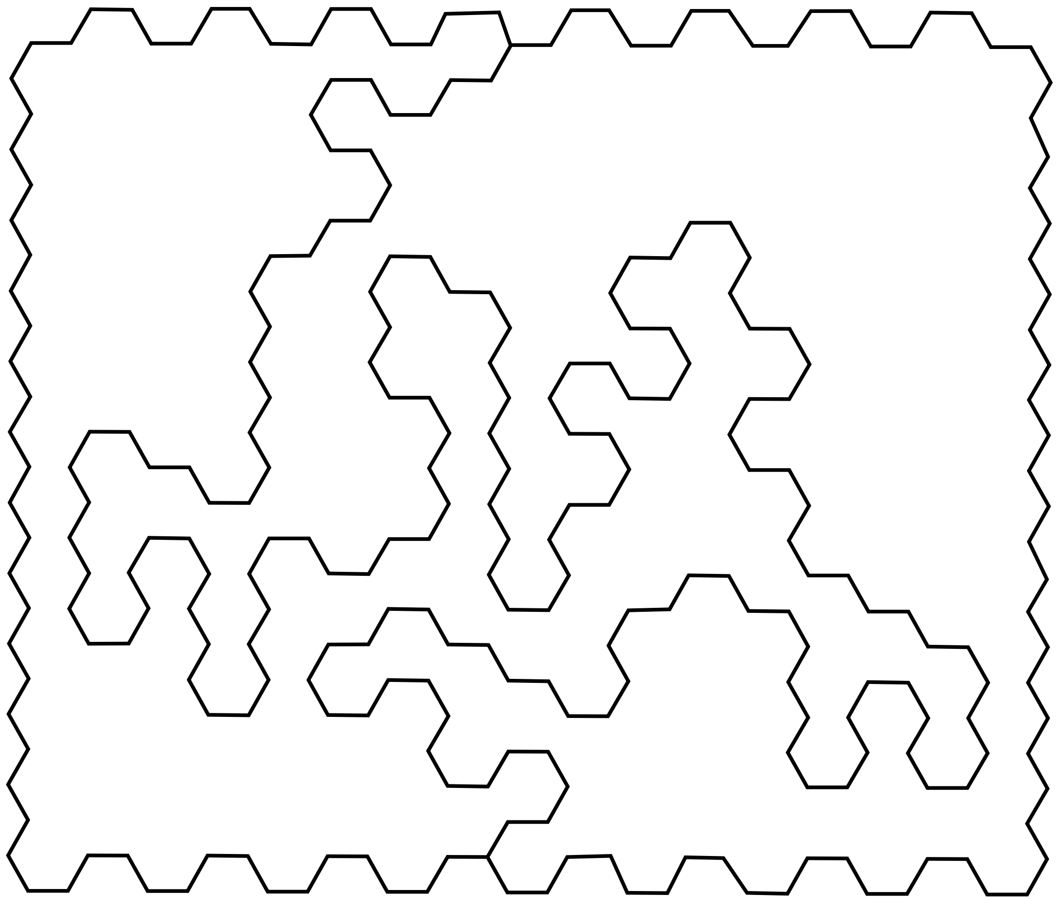
A percolation configuration with the imposed boundary conditions



The result of "swallowing the islands"



The percolation exploration path



Theorem. Let (D, a, b) be a Jordan domain with two distinct selected points on its boundary ∂D, and suppose that $D^\delta \subset \delta\mathcal{H}$ are Jordan sets with two distinct selected e-vertices a^δ, $b^\delta \in \partial D^\delta$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ as $\delta \to 0$.

If $\gamma^\delta_{D,a,b}$ denotes the percolation exploration path inside D^δ from a^δ to b^δ, then $\gamma^\delta_{D,a,b}$ converges in distribution as $\delta \downarrow 0$ to $\gamma_{D,a,b}$, the trace of chordal SLE$_6$ inside D from a to b

Elements of the proof

There are essentially two main parts to the proof. The first is a characterization of SLE_6, and the second is the fact that any subsequential limit of the exploration path satisfies this characterization.

The actual proof of the theorem is relatively short *once all of the preliminary lemmas and preparatory theorems have been established*.

The Proof!

Proof. Consider $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ and the percolation exploration path $\gamma^\delta_{D,a,b}$. The law of $\gamma^\delta_{D,a,b}$ is a distribution on curves. An earlier result of Aizenman and Burchard [1] (in particular, Theorem A.1 in Appendix A.1) is that this family $\gamma^\delta_{D,a,b}$ converges in distribution along subsequential limits $\delta_k \downarrow 0$ to the law of some curve γ.

Since the filling of any subsequential limit

$$\tilde{\gamma} = \tilde{\gamma}_{D,a,b} = \lim_{\delta_k \downarrow 0} \gamma^{\delta_k}_{D,a,b}$$

satisfies the spatial Markov property (Theorem C) and the hitting distribution of $\tilde{\gamma}$ is determined by Cardy's formula (Theorem A), it follows from Theorem B that the limit is unique and that the law of $\gamma^\delta_{D,a,b}$ converges to the law of the trace $\gamma_{D,a,b}$ of chordal SLE_6 inside D from a to b as $\delta \to 0$. $\square$

Of course, we now need to explain the different elements of the proof!

Historical remark

A beautiful argument due to Schramm showed that if the scaling limit of the exploration path exists and is conformally invariant, then it must be SLE_κ for some κ. The value $\kappa = 6$ is then obtained by noting that Cardy's formula is satisfied only by SLE_6. The proof of this result was announced by Smirnov in 2001 [13], although a detailed proof of convergence did not appear until 2005. The work by Camia and Newman [6] presents that proof in an essentially self-contained form. We also mention that convergence of the exploration path to SLE_6 was used by Smirnov and Werner [14]; and Lawler, Schramm, and Werner [9] to rigorously derive the values of various percolation critical exponents. Camia and Newman also used the convergence to obtain the full scaling limit of critical percolation in two dimensions. Lectures by P. Nolin and C. Hongler will discuss these critical exponents and the full scaling limit, respectively.

The metric space of curves

One of the first tasks that needs to be done in order to prove that the percolation exploration path converges to the trace of chordal SLE_6 is to state precisely the metric space of curves that will be considered in order to discuss weak convergence of the appropriate measures.

The approach followed by Camia and Newman [6] in dealing with the scaling limit is the one discussed by Aizenman and Burchard [1].

Suppose that $\Lambda \subset \mathbb{C}$ is a closed, bounded subset of $\mathbb{C}$, and denote by $\mathcal{S}_\Lambda$ the set of continuous curves $\gamma : [0, t_\gamma] \to \Lambda$. (If $t_\gamma = \infty$, then we let $\gamma(\infty) = \lim_{t\to\infty} \gamma(t)$.) Define the metric

$$\mathrm{d}(\gamma_1, \gamma_2) \equiv \inf_{\psi} \sup_{0 \le s \le t_{\gamma_1}} |\gamma_1(s) - \gamma_2(\psi(s))| \tag{1}$$

where the infimum is over all increasing homeomorphisms $\psi : [0, t_{\gamma_1}] \to [0, t_{\gamma_2}]$.

The metric space of curves

A theorem of Prohorov shows that convergence of a sequence of probability measures in the Prohorov metric space $(\mathcal{M}, \wp)$ is equivalent to weak convergence of the same probability measures, and the Portmanteau theorem gives several other conditions equivalent to weak convergence.

The following fact which relates convergence in d to convergence in $\wp$ is both easy to prove and extremely useful for establishing weak convergence.

Proposition. Consider the metric space $(\mathcal{S}_\Lambda^*, \mathrm{d})$, and let γ_1, γ_2 be $(\mathcal{S}_\Lambda^*, \mathrm{d})$-valued random variables. If $\mathbf{P}\{\mathrm{d}(\gamma_1, \gamma_2) \geq \varepsilon\} \leq \varepsilon$, then $\wp(\mathcal{L}(\gamma_1), \mathcal{L}(\gamma_2)) \leq \varepsilon$ where $\mathcal{L}(\gamma_i)$ denotes the law of γ_i, $i = 1, 2$.

The metric space of curves

In order to simplify notation, Camia and Newman [6] write $\mathcal{S}_\Lambda$ for this metric space $(\mathcal{S}^*_\Lambda, \mathrm{d})$ of curves regarded as equivalence classes of continuous functions from the unit interval to $\mathbb{C}$ modulo monotonic reparametrizations. They also write γ to represent a particular curve and $\gamma(t)$ for a parametrization of γ.

If $\mathbb{P}$ is the configurational measure for i.i.d. Bernoulli$(1/2)$ site percolation on $\mathcal{T}$, then $\mathbb{P}$ induces a probability measure $\mu^\delta_{D,a,b}$ on exploration paths $\gamma^\delta_{D,a,b}$ inside D^δ from a^δ to b^δ.

The convergence theorem states that as $\delta \downarrow 0$, these measures converge weakly with respect to the metric d to $\mu_{D,a,b}$, the law of $\gamma_{D,a,b}$, the trace of chordal SLE_6 inside D from a to b.

Riemann mapping theorem

If D and D' are two simply connected domains, we will sometimes write $\mathcal{C}(D, D')$ to denote the set of all conformal transformations from D onto D'.

Riemann Mapping Theorem. Suppose that D is a simply connected proper subsets of $\mathbb{C}$, and let $z_0 \in D$. Then there exists a unique conformal transformation f of $\mathbb{D}$ onto D satisfying $f(0) = z_0$, $f'(0) > 0$.

Continuity Theorem. If D is a domain whose boundary ∂D is locally connected and $f : \mathbb{D} \to D$ is a conformal transformation, then f can be extended continuously to a map of $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$ to the closed disk $\overline{D}$.

Carathéodory Extension Theorem. If D is a domain bounded by a Jordan curve ∂D, and $f : \mathbb{D} \to D$ is a conformal transformation, then f can be extended to a homeomorphism of $\mathbb{D}$ onto the closed disk $\overline{\mathbb{D}}$.

Carathéodory convergence

In many of the examples where a discrete process is proved to converge to a continuous process, the continuous process is described by a family of measures parametrized by (among other things) domains $D \subset \mathbb{C}$. The discrete process is defined on a lattice-type approximation to D in such a way that as the lattice spacing shrinks to 0, the lattice approximation converges to D. The convergence of domains is often in the Carathéodory sense.

Let $\mathcal{D}$ denote the set of simply connected Jordan domains containing the origin.

Carathéodory Convergence Theorem. Suppose that D_n is a sequence of domains with $D_n \in \mathcal{D}$ for each n, and let $f_n \in \mathcal{C}(\mathbb{D}, D_n)$ with $f_n(0) = 0$, $f_n'(0) > 0$. Suppose further that $D \in \mathcal{D}$ and $f \in \mathcal{C}(\mathbb{D}, D)$ with $f(0) = 0$, $f'(0) > 0$. Then $f_n \to f$ uniformly on compacta of $\mathbb{D}$ if and only if $D_n \stackrel{\text{Cara}}{\to} D$.

It must be noted that the Carathéodory convergence theorem is not enough for the work of Camia and Newman. Instead of delving into these ideas more deeply, we refer the interested reader to Appendix A of [6] for a thorough discussion of the required extensions.

We now give an example to show how one might combine Carathéodory convergence with convergence in the metric d.

Proposition. Suppose that F_n, F are conformal mappings of the unit disk $\mathbb{D}$ and that $F_n \to F$ uniformly on compacta of $\mathbb{D}$. If $K \subset \mathbb{D}$ is compact and $\gamma \in \mathcal{S}_K$, then $\mathrm{d}(F_n \circ \gamma, F \circ \gamma) \to 0$ as $n \to \infty$.

Cardy's formula

Let D be a bounded simply connected domain containing the origin whose boundary ∂D is a continuous curve.

Let $\varphi : \mathbb{D} \to D$ be the unique conformal map from the unit disc $\mathbb{D}$ to D with $\varphi(0) = 0$ and $\varphi'(0) > 0$ whose existence is guaranteed by the Riemann mapping theorem.

By the continuity theorem, φ has a continuous extension to $\overline{\mathbb{D}}$.

(If D is a Jordan domain, then the Carathéodory extension theorem implies this extension is also injective.)

Let z_1, z_2, z_3, z_4 be four points of ∂D ordered counterclockwise, and let w_1, w_2, w_3, w_4 be the corresponding four points of $\partial\mathbb{D}$ ordered counterclockwise with $z_j = \varphi(w_j)$.

Cardy's formula

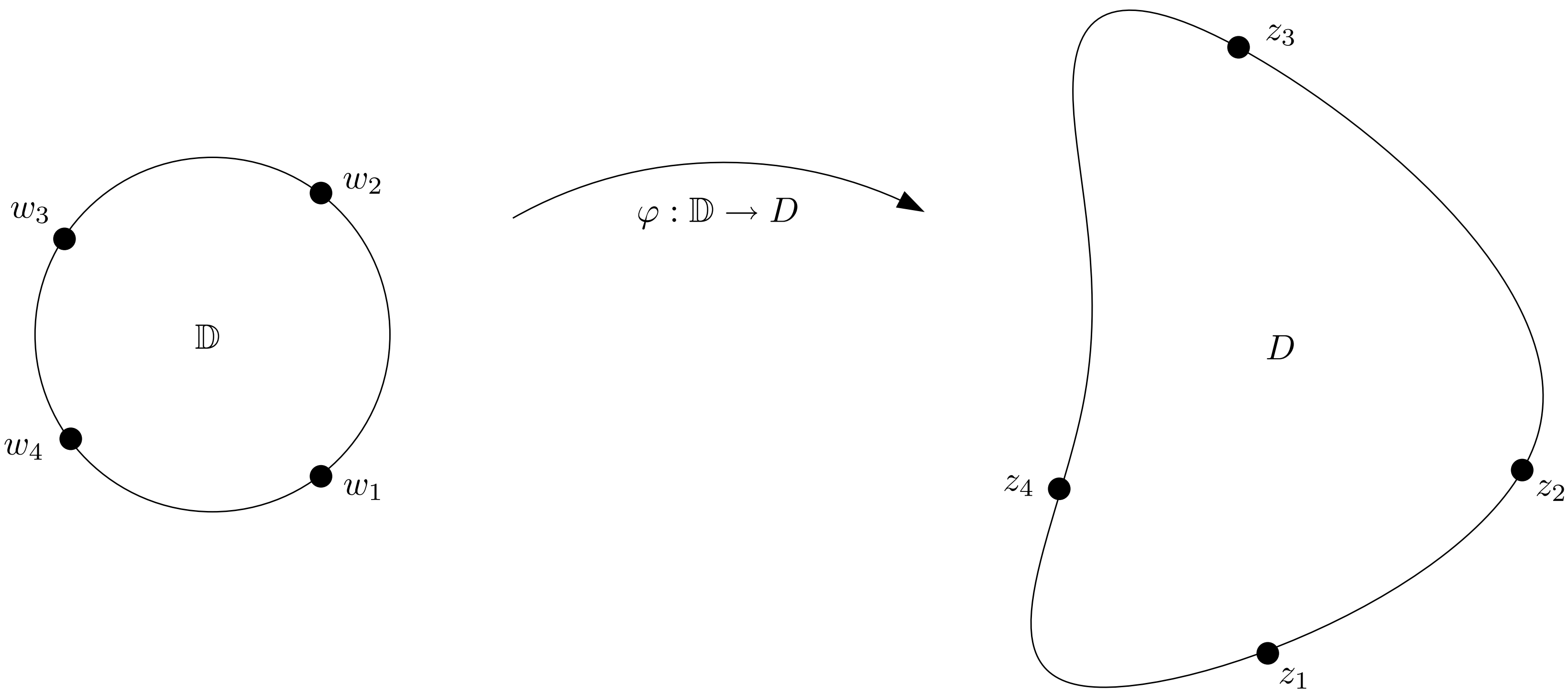


Figure 1: The map $\varphi : \mathbb{D} \to D$ in the setup for Cardy's formula.

Cardy's formula

Cardy's formula [5] for the probability $\Phi_D(z_1, z_2; z_3, z_4)$ of an "open crossing" in D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is

$$\Phi_D(z_1, z_2; z_3, z_4) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta), \tag{2}$$

where ${}_2F_1$ is a hypergeometric function and

$$\eta = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}$$

denotes the *cross ratio*.

Carleson's observation for Cardy's formula

The following observation was first made by L. Carleson. Using properties of the hypergeometric function one can write

$$\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} z^{1/3} {}_2F_1(1/3, 2/3; 4/3; z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

Furthermore, the function

$$z \mapsto \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

is the Schwarz-Christoffel transformation of $\mathbb{H}$ onto the equilateral traingle sending $0 \mapsto 0$, $1 \mapsto 1$, and $\infty \mapsto (1+i\sqrt{3})/2$.

Hence, if D is this equilateral triangle, then Cardy's formula takes the particularly nice form

$$\Phi_D(1, (1+i\sqrt{3})/2); 0, x) = x.$$

Cardy's formula

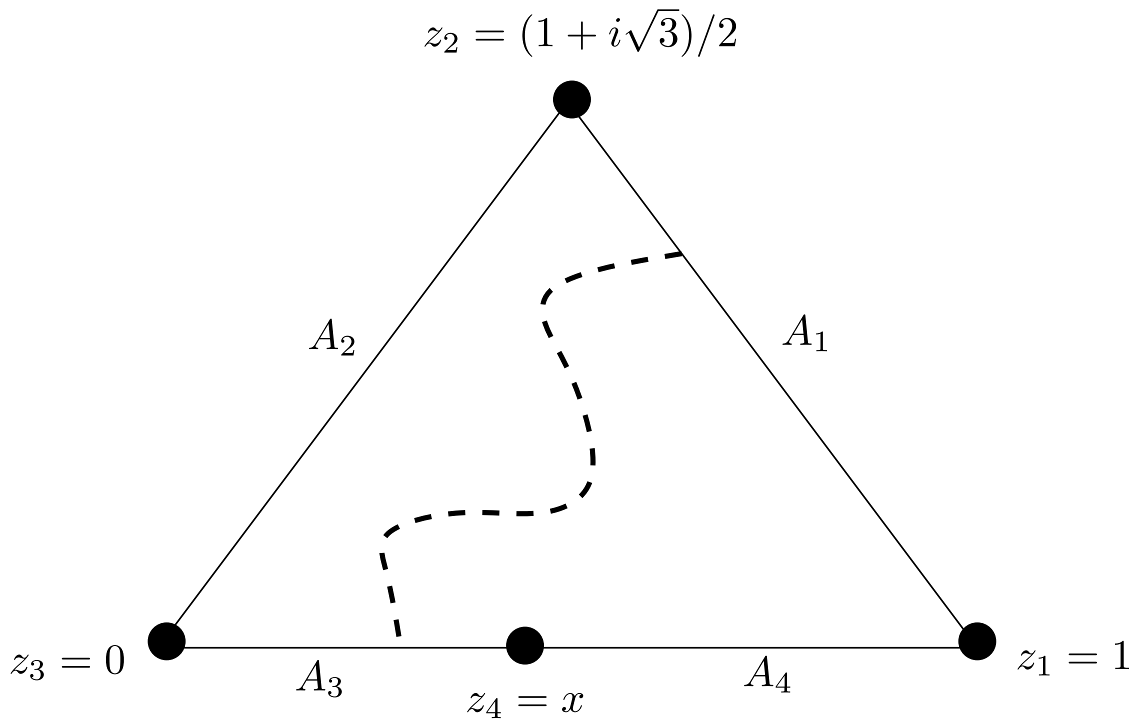


Figure 2: Cardy's formula for an equilateral triangle.

Smirnov's theorem

The following important result was announced by Smirnov [13] in 2001.

A complete and thorough proof may be found in [3].

Theorem. Let D be a Jordan domain whose boundary ∂D is a finite union of smooth (e.g., C^2) curves.

As $\delta \to 0$, the limit of the probability of an open crossing inside D from the counterclockwise arc $\overline{z_1^\delta z_2^\delta}$ to the counterclockwise arc $\overline{z_3^\delta z_4^\delta}$ exists, is a conformal invariant of (D, z_1, z_2, z_3, z_4), and is given by Cardy's formula $\Phi_D(z_1, z_2; z_3, z_4)$.

In order to use Smirnov's theorem for the proof of convergence of the exploration path to SLE$_6$, a slightly extended version was required by Camia and Newman [6]. They proved Theorem A which extends Smirnov's theorem to a larger class of domains (including all Jordan domains).

Admissible domains

Suppose that D is a bounded, simply connected domain whose boundary ∂D is a continuous curve.

Let a, c, d be three points of ∂D (technically, three prime ends of D) in counterclockwise order.

We say that D is *admissible with respect to* (a, c, d) if

(i) the counterclockwise arcs $\overline{da}$, $\overline{ac}$, and $\overline{cd}$ are simple curves;

(ii) the arc $\overline{cd}$ does not touch the interior of either of the other two arcs; and

(iii) from each point in $\overline{cd}$ there is a path to infinity that does not cross ∂D.

Note that if D is Jordan, then D is admissible for any three counterclockwise points $a, c, d \in \partial D$.

An example of a non-Jordan, admissible domain

However, there are domains which arise naturally in the proofs of theorems in [6] that are not Jordan, but are admissible.

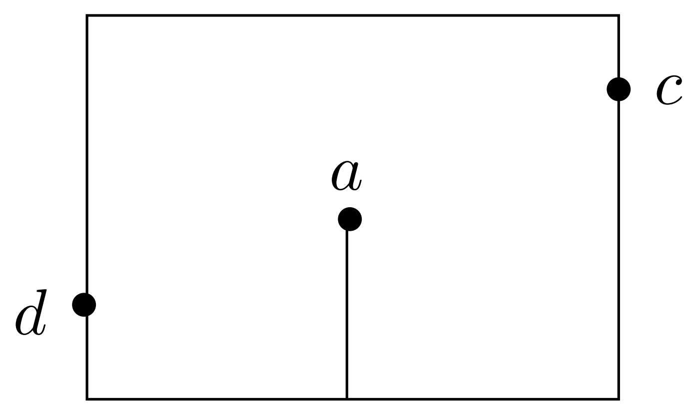


Figure 3: An example of an admissible, non-Jordan domain.

A stronger version of Cardy's formula

Theorem A. Consider a sequence $\{(D_k, a_k, c_k, b_k, d_k)\}$ of domains D_k containing the origin, admissible with respect to the points a_k, c_k, d_k on ∂D_k, and with b_k belonging to the interior of the counterclockwise arc $\overline{c_k d_k}$ of ∂D_k. Assume that, as $k \to \infty$, $b_k \to b$ and there is convergence in the metric d of the counterclockwise arcs $\overline{d_k a_k}$, $\overline{a_k c_k}$, $\overline{c_k d_k}$ to the corresponding counterclockwise arcs $\overline{da}$, $\overline{ac}$, $\overline{cd}$ of ∂D, where D is a domain containing the origin, admissible with respect to (a, c, d), and b belongs to the interior of $\overline{cd}$. Then, for any sequence $\delta_k \downarrow 0$, the probability $\Phi^{\delta_k}_{D_k}(a_k, c_k; b_k, d_k)$ of an open crossing inside D_k from $\overline{a_k c_k}$ to $\overline{b_k d_k}$ converges as $k \to \infty$ to Cardy's formula $\Phi_D(a, c; b, d)$ for an open crossing inside D from $\overline{ac}$ to $\overline{bd}$ as given by (2).

Characterization of SLE_6

Suppose that D is a simply connected domain whose boundary is a continuous curve, and let a, $b \in \partial D$ be distinct.

Suppose further that $\mu_{D,a,b}$ is a probability measure on continuous curves $\gamma = \gamma_{D,a,b} : [0,\infty] \to \overline{D}$ with $\gamma(0) = a$ and $\gamma(\infty) = b$. That is, $\mathcal{L}(\gamma_{D,a,b}) = \mu_{D,a,b}$.

Let $D_t = D \setminus K_t$ denote the unique connected component of $D \setminus \gamma[0,t]$ whose closure contains b.

This implicitly defines K_t, the **filling** of $\gamma[0,t]$, which is a closed connected subset of $\overline{D}$. We call K_t a **hull** if $\overline{K_t \cap D} = K_t$.

Note: If γ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then $K_t = \gamma[0,t]$ if $0 < \kappa \leq 4$, but $K_t \neq \gamma[0,t]$ for $4 < \kappa < 8$ since in this case $\gamma(0,t] \cap \mathbb{R} \neq \emptyset$.

Let $E \subset D$ be a closed subset of $\overline{D}$ such that $a \notin E$, $b \in E$, and $D' = D \setminus E$ is a bounded simply connected domain whose boundary is a continuous curve containing the counterclockwise arc $\overline{cd}$ that does not belong to ∂D (except for its endpoints c and d – see Figure 4).

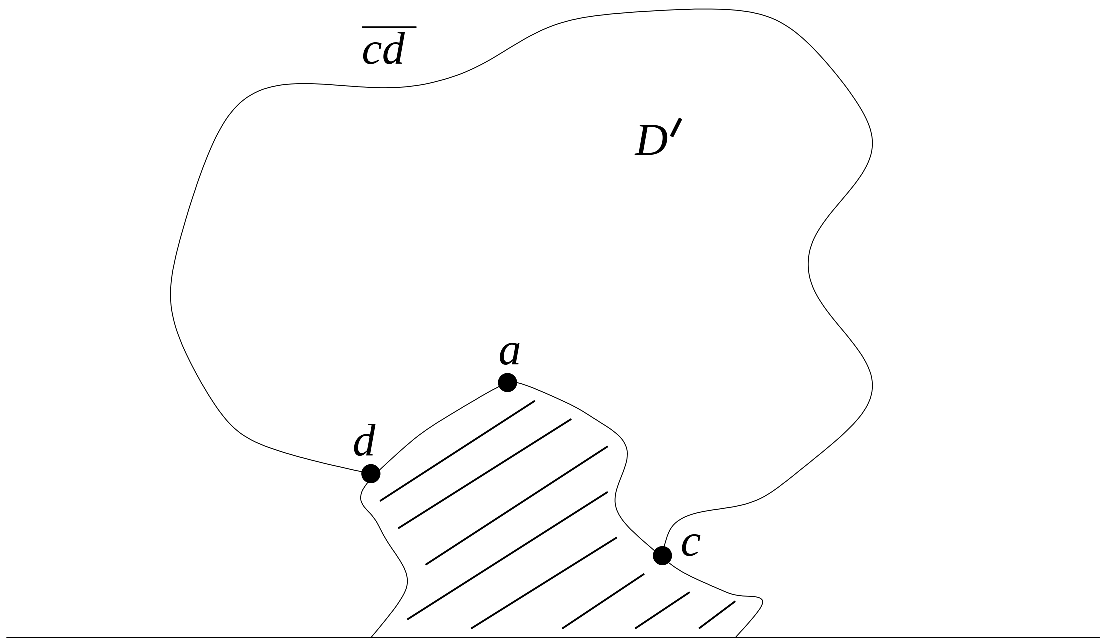


Figure 4: D is the upper half-plane $\mathbb{H}$ with the shaded portion removed, $b = \infty$, E is an unbounded subdomain, and $D' = D \setminus E$ is indicated in the figure. The counterclockwise arc $\overline{cd}$ indicated in the figure belongs to $\partial D'$.

Let $T = \inf\{t : K_t \cap E \neq \emptyset\}$ be the first time that $\gamma(t)$ hits E. We say that the **hitting distribution** of $\gamma(t)$ at the stopping time T **is determined by Cardy's formula** if, for any E and any counterclockwise arc $\overline{xy}$ of $\overline{cd}$, the probability that γ hits E at time T on $\overline{xy}$ is given by

$$\mathbf{P}(\gamma(T) \in \overline{xy}) = \Phi_{D'}(a, c; x, d) - \Phi_{D'}(a, c; y, d). \tag{3}$$

Now let f_0 be a conformal map from the upper half-plane $\mathbb{H}$ to D such that ${f_0}^{-1}(a) = 0$ and ${f_0}^{-1}(b) = \infty$.

These two conditions determine f_0 only up to a scaling factor.

For $\varepsilon > 0$ fixed, let $C(\varepsilon) = \{z : |z| < \varepsilon\} \cap \mathbb{H}$ denote the semi-ball of radius ε centered at 0.

Let $T_1 = T_1(\varepsilon)$ denote the first time $\gamma(t)$ hits $D \setminus G_1$, where $G_1 \equiv f_0(C(0, \varepsilon))$.

Define recursively T_{j+1} as the first time $\gamma[T_j, \infty)$ hits $D_{T_j} \setminus G_{j+1}$, where $D_{T_j} \equiv D \setminus K_{T_j}$, $G_{j+1} \equiv f_{T_j}(C(0, \varepsilon))$, and f_{T_j} is a conformal map from $\mathbb{H}$ to D_{T_j} whose inverse maps $\gamma(T_j)$ to 0 and b to ∞.

We choose f_{T_j} so that its inverse is the composition of the restriction of ${f_0}^{-1}$ to D_{T_j} with φ_{T_j} where $\varphi_{T_j} : \mathbb{H} \setminus {f_0}^{-1}(K_{T_j}) \to \mathbb{H}$ is the unique conformal transformation with $\varphi'_{T_j}(\infty) = 1$, and sending $\infty \mapsto \infty$ and ${f_0}^{-1}(\gamma(T_j)) \mapsto 0$.

Notice that G_{j+1} is a bounded simply connected domain chosen so that the conformal transformation which maps D_{T_j} to $\mathbb{H}$ maps G_{j+1} to the semi-ball $C(0, \varepsilon)$ centred at the origin on the real line.

With these definitions, consider the (discrete-time) stochastic process $X_j \equiv (K_{T_j}, \gamma(T_j))$ for $j = 1, 2, \ldots$. We say that K_t satisfies the **spatial Markov property** if each K_{T_j} is a hull and X_j for $j = 1, 2, \ldots$ is a Markov chain (for any choice of the map f_0).

Note. The conformal invariance and Markovian properties of SLE_6 imply that the hull of chordal SLE_6 satisfies the spatial Markov property.

Theorem B. If the filling process $\{K_t, t \geq 0\}$ of a continuous curve $\gamma_{D,a,b}$ satisfies the spatial Markov property and its hitting distribution is determined by Cardy's formula, then $\gamma_{D,a,b}$ is distributed like the trace of chordal SLE_6 in D from a to b.

Convergence of the exploration path

The main technical difficulty in the approach of Camia and Newman [6] is to obtain a Markov property for any scaling limit of the percolation exploration path.

The difficulty is that in the scaling limit, the exploration path touches itself and the boundary of the domain infinitely often.

The standard percolation bound on multiple crossings of a "semi-annulus" only applies to the case of a "flat" boundary.

Camia and Newman resolve the issue by proving that Cardy's formula is continuous with respect to changes in the domain.

As a final note, we remark that this technical issue is somewhat surprising since an even stronger Markov property trivially holds for the exploration path itself.

Theorem C. If $\tilde{\gamma}$ is any subsequential limit of the percolation exploration path $\gamma^{\delta}_{D,a,b}$, then $\tilde{K}_t$, the filling process of $\tilde{\gamma}[0,t]$, satisfies the spatial Markov property.

References

[1] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99:419–453, 1999.

[2] H. Bergström. *Weak Convergence of Measures*. Academic Press, New York, NY, 1982.

[3] B. Bollobás and O. Riordan. *Percolation*. Cambridge University Press, Cambridge, UK, 2006.

[4] V. S. Borkar. *Probability Theory: An Advanced Course*. Springer-Verlag, New York, NY, 1995.

[5] J.L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25:L201–L206, 1992.

[6] F. Camia and C.M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. *Probab. Theory Related Fields*, 139:473–519, 2007.

[7] P. L. Duren. *Univalent Functions.* Springer-Verlag, New York, NY, 1983.

[8] M. J. Kozdron. On the scaling limit of simple random walk excursion measure in the plane. *ALEA Lat. Am. J. Probab. Math. Statist.*, 2:125–155, 2006.

[9] G. Lawler, O. Schramm and W. Werner. One arm exponent for critical 2D percolation. *Electron. J. Probab.*, 7:1–13 (paper no. 2), 2002.

[10] G. F. Lawler and W. Werner. Universality for conformally invariant intersection exponents. *J. Eur. Math. Soc.*, 2:291–328, 2000.

[11] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps.* Springer-Verlag, New York, NY, 1992.

[12] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*,118:221–288, 2000.

[13] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333:239–244, 2001.

[14] S. Smirnov and W. Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8:729–744, 2001.

[15] W. Werner. Lectures on two-dimensional critical percolation. *Lectures given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics.* Preprint, 2007. Available online at `arxiv:math.PR/0710.0856`.