Convergence of 2D critical percolation to SLE_6

Lecture given at the Mathematisches Forschungsinstitut Oberwolfach (MFO) during the *Arbeitsgemeinschaft: Percolation* held October 8–13, 2007

Michael J. Kozdron*
Department of Mathematics & Statistics, College West 307.14
University of Regina, Regina, SK S4S 0A2 Canada
kozdron@stat.math.uregina.ca

October 11, 2007

Abstract

In this lecture we present the main ideas of the convergence, in the scaling limit, of the critical site percolation exploration path on the triangular lattice to SLE_6. This example is one of only a select few where convergence to SLE of an appropriate discrete statistical mechanics model is completely understood. As such, the result is crucial for the determination of the critical exponents in two dimensions and in all applications of SLE to percolation. Our primary reference for this result is the recent paper of Camia and Newman [6].

2000 Mathematics Subject Classification. 82B27, 60K35, 82B43, 60D05, 30C35.

Key words and phrases. Schramm-Loewner evolution, percolation, continuum scaling limit, triangular lattice, hexagonal lattice, conformal invariance, Cardy's formula, percolation exploration process.

1 Introductory remarks and a precise statement of the main result

The goal of this lecture is to explain the convergence of the critical site percolation exploration path on the triangular lattice to the trace of chordal SLE_6 for Jordan domains. Our primary reference is the recent paper by Camia and Newman [6] and we cite many results verbatim from that work. The present author makes no claims of originality, but it is hoped that by highlighting some key elements of the proof in a slightly different way than is done in [6], the interested reader can use this present work as a companion to help increase his or her understanding of [6].

It should be noted that there is a recent preprint by W. Werner [15] containing lecture notes from a short course given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics. Lecture 3 in those notes is concerned with a proof of this convergence result, but Werner follows a different approach than Camia and Newman. In fact, Werner's notes [15] contain six lectures and a set of exercises on critical site percolation on the triangular lattice that coincide with the topic of this Arbeitsgemeinschaft; we highly recommend reading them.

The primary theorem that we will be concerned with is the following precise formulation of the convergence of the percolation exploration path to SLE_6 as given by Theorem 1.1 below. In order to explain the statement of the theorem, however, we will need to introduce some notation. Although everything will not be quite as precise as it should be, such simplifications (in our opinion) allow us to capture the spirit of the theorem (and its proof).

Let $\mathcal{T}$ denote the standard two-dimensional triangular lattice with lattice spacing 1, and let $\mathcal{H}$ denote the hexagonal lattice which is dual to $\mathcal{T}$. For $\delta > 0$, we write $\delta\mathcal{H}$ to denote the hexagonal lattice with lattice spacing δ; see Figure 1.

*Research supported in part by the Natural Sciences and Engineering Research Council of Canada.

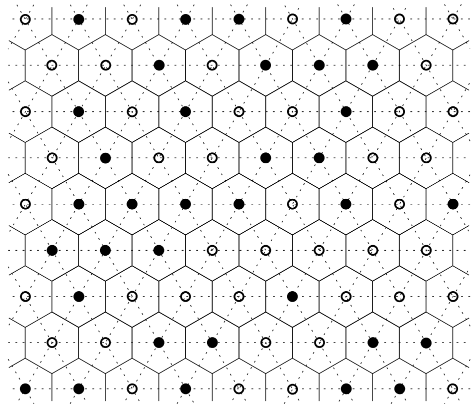

Figure 1: A portion of the triangular lattice $\mathcal{T}$ and its dual lattice $\mathcal{H}$. Note that site percolation on $\mathcal{T}$ corresponds to face percolation on $\mathcal{H}$.

Let $D \subset \mathbb{C}$ be a bounded, simply connected Jordan domain. That is, D is a simply connected domain whose boundary ∂D is a Jordan curve (i.e., ∂D is a simple closed curve which is homeomorphic to the unit circle).

Suppose that $D^\delta \subset \delta\mathcal{H}$ is a Jordan set which approximates D. That is, D^δ is a simply connected subset of the hexagonal lattice with lattice spacing δ whose external site boundary is a simple closed loop of hexagons such that D^δ is a discrete approximation to D. Suppose further that a, $b \in \partial D$ are distinct boundary points, and let a^δ, $b^\delta \in \partial D^\delta$ be the corresponding external boundary vertices (or e-vertices). Without being more precise about this exact approximation, we denote by (D, a, b) the simply connected Jordan domain with two distinguished boundary points, and let its δ-scale approximation be denoted by $(D^\delta, a^\delta, b^\delta)$. Essentially, we think of choosing $D^\delta \equiv D \cap \delta\mathcal{H}$. (But this may not produce a simply connected D^δ so we do need to be careful.) A bit more technically, we assume that D^δ, a^δ, and b^δ are chosen so that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ in the Carathéodory sense as $\delta \to 0$.

If we now consider D^δ with distinguished e-vertices a^δ and b^δ, then we can see that these two distinguished boundary points partition the (topological) boundary of D^δ into two disjoint arcs. Associate to all external boundary hexagons on one of the arcs the colour "red" and associate to all boundary hexagons on the other arc the colour "white." Perform critical site percolation on D^δ; that is, for each remaining interior hexagon colour it either red with probability 1/2 or white with probability 1/2. There will be a resulting *interface* joining a^δ with b^δ; that is, a simple path connecting a^δ to b^δ with the property that all hexagons on one side of the path will be white while all hexagons on the other side of the path will be red. We call this path/interface the (critical site) percolation exploration path and denote it by $\gamma^\delta_{D,a,b}$. As $\delta \downarrow 0$, it is this path that converges to chordal SLE_6 in D from a to b.

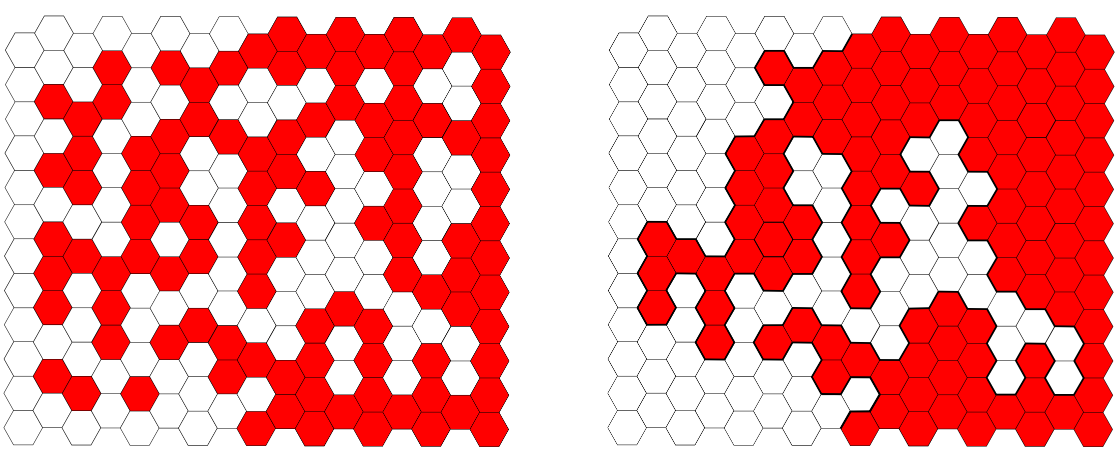

Figure 2: On the left is the realization of the percolation configuration with the imposed boundary conditions, and on the right is the result of "swallowing the islands."

Figure 2 shows schematically one way of producing the percolation exploration path. Given the realization of the percolation configuration with the boundary conditions (as shown on the left of Figure 2) we then "swallow any islands" by swapping the colour of an "island" with the colour of the "ocean" surrounding it (as shown on the right of Figure 2). This produces two disjoint sets—one coloured red and the other coloured white. The percolation exploration path is exactly the interface between these two sets. If we now delete all of the hexagons, then what remains is the percolation exploration path as shown in Figure 3.

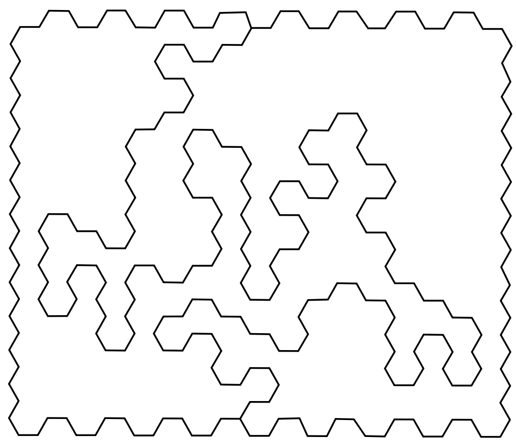

Figure 3: The exploration path resulting from the percolation configuration in Figure 2.

Theorem 1.1 (Camia and Newman [6]). *Let (D, a, b) be a Jordan domain with two distinct selected points on its boundary ∂D, and suppose that $D^\delta \subset \delta\mathcal{H}$ are Jordan sets with two distinct selected e-vertices a^δ, $b^\delta \in \partial D^\delta$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ as $\delta \to 0$. If $\gamma^\delta_{D,a,b}$ denotes the percolation exploration path inside D^δ from a^δ to b^δ, then $\gamma^\delta_{D,a,b}$ converges in distribution as $\delta \downarrow 0$ to $\gamma_{D,a,b}$, the trace of chordal SLE$_6$ inside D from a to b*

There are essentially two main parts to the proof. The first is a characterization of SLE$_6$, and the second is the fact that any subsequential limit of the exploration path satisfies this characterization.

The actual proof of the theorem is relatively short *once all of the preliminary lemmas and preparatory theorems have been established.*

Proof. Consider $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ and the percolation exploration path $\gamma^\delta_{D,a,b}$. The law of $\gamma^\delta_{D,a,b}$ is a distribution on curves. An earlier result of Aizenman and Burchard [1] (in particular, Theorem A.1 in Appendix A.1) is that this family $\gamma^\delta_{D,a,b}$ converges in distribution along subsequential limits $\delta_k \downarrow 0$ to the law of some curve γ.

Since the filling of any subsequential limit

$$\tilde{\gamma} \equiv \tilde{\gamma}_{D,a,b} \equiv \lim_{\delta_k \downarrow 0} \gamma^{\delta_k}_{D,a,b}$$

satisfies the spatial Markov property (Theorem 5.1) and the hitting distribution of $\tilde{\gamma}$ is determined by Cardy's formula (Theorem 3.2), it follows from Theorem 4.1 that the limit is unique and that the law of $\gamma^\delta_{D,a,b}$ converges as $\delta \to 0$ to the law of $\gamma_{D,a,b}$, the trace of chordal SLE$_6$ in D from a to b. ☐

Of course, we now need to explain the different elements of the proof!

Remark. As a historical note, we mention that a beautiful argument due to Schramm [12] showed that if the scaling limit of the exploration path exists and is conformally invariant, then it must be SLE$_\kappa$ for some κ. The value $\kappa = 6$ is then obtained by noting that Cardy's formula is satisfied only by SLE$_6$. The proof of this result was announced by Smirnov in 2001 [13], although a detailed proof of convergence did not appear until 2005. The work by Camia and Newman [6] presents that proof in an essentially self-contained form. We also mention that convergence of the exploration path to SLE_6 was used by Smirnov and Werner [14]; and Lawler, Schramm, and Werner [9] to rigorously derive the values of various percolation critical exponents. Camia and Newman also used the convergence to obtain the full scaling limit of critical percolation in two dimensions. Lectures by P. Nolin and C. Hongler during the current Arbeitsgemeinschaft will discuss these critical exponents and the full scaling limit, respectively.

2 Background material

Two of the technical matters that need to be handled in proving Theorem 1.1 are to precisely state the topology of weak convergence of measures on curves (i.e., the exploration path converging to the trace of chordal SLE_6) and to explain how the domain $D \subset \mathbb{C}$ will be approximated by discrete subsets. There are "standard" ways to deal with such things and the purpose of this section on background material is to provide the reader with some of the basic facts regarding the metric space of curves given by [1] and Carathéodory convergence of domains.

2.1 The metric space of curves

One of the first tasks that needs to be done in order to prove that the percolation exploration path converges to the trace of chordal SLE_6 is to state precisely the metric space of curves that will be considered in order to discuss weak convergence of the appropriate measures. Most of this material on metric spaces of curves and on weak convergence of measures is usually considered "standard," and is therefore not always discussed explicitly. References where such standard details may be found include [2] and [4]. The approach followed by Camia and Newman [6] in dealing with the scaling limit is the one discussed by Aizenman and Burchard [1] which we now review.[1]

Suppose that $\Lambda \subset \mathbb{C}$ is a closed, bounded subset of $\mathbb{C}$, and denote by $\mathcal{S}_\Lambda$ the set of continuous curves $\gamma : [0, t_\gamma] \to \Lambda$. (If $t_\gamma \equiv \infty$, then we let $\gamma(\infty) = \lim_{t\to\infty} \gamma(t)$.) Define the metric

$$\mathrm{d}(\gamma_1, \gamma_2) \equiv \inf_{\psi} \sup_{0 \le s \le t_{\gamma_1}} |\gamma_1(s) - \gamma_2(\psi(s))| \tag{1}$$

where the infimum is over all increasing homeomorphisms $\psi : [0, t_{\gamma_1}] \to [0, t_{\gamma_2}]$.

Call $\tilde{\gamma}$ a *reparameterization* of $\gamma \in \mathcal{S}_\Lambda$ with parameterization ψ if $\psi : [0, t_\gamma] \to [0, t_{\tilde{\gamma}}]$ is an increasing homeomorphism such that $\gamma(t) = \tilde{\gamma}(\psi(t))$ for each $0 \le t \le t_\gamma$. If $\tilde{\gamma}$ is a reparameterization of γ under ψ, then γ is a reparameterization of $\tilde{\gamma}$ under ψ^{-1}, and we write $\gamma \overset{\text{par}}{\sim} \tilde{\gamma}$. Finally, let $\mathcal{S}^*_\Lambda$ be the set of equivalence classes of curves $\gamma \in \mathcal{S}_\Lambda$ under the relation $\overset{\text{par}}{\sim}$, so that the metric d identifies curves which are equal modulo time reparameterization. In fact, $(\mathcal{S}^*_\Lambda, \mathrm{d})$ is a complete separable metric space; see [1, Lemma 2.1].

Given the metric space $(\mathcal{S}^*_\Lambda, \mathrm{d})$, let $\mathcal{B} \equiv \mathcal{B}_\mathrm{d}(\Lambda)$ denote the Borel σ-algebra associated to the topology induced by d so that $(\mathcal{S}^*_\Lambda, \mathcal{B})$ is a measurable space. Let $\mathcal{M} \equiv \mathcal{M}(\mathcal{S}^*_\Lambda)$ denote the space of probability measures on $(\mathcal{S}^*_\Lambda, \mathcal{B})$. The Prohorov metric space $(\mathcal{M}, \wp)$ is itself a complete, separable metric space where the metric $\wp$ is given by the following.

Definition 2.1. Let (Ξ, ρ) be a metric space. If $m_1, m_2 \in \mathcal{M} = \mathcal{M}(\Xi)$, let $\wp : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ denote the *Prohorov metric* given by $\wp(m_1, m_2) \equiv \inf\{\varepsilon > 0 : m_1(F) \le m_2(F^{(\varepsilon)}) + \varepsilon,\ \ m_2(F) \le m_1(F^{(\varepsilon)}) + \varepsilon\ \ \forall\, F \in \mathcal{B}_\rho\}$ where $F^{(\varepsilon)} \equiv \{x \in \Xi : \rho(x, y) < \varepsilon \text{ for some } y \in F\}$.

A theorem of Prohorov shows that convergence of a sequence of probability measures in the Prohorov metric space is equivalent to weak convergence of the same probability measures, and the Portmanteau theorem gives several other conditions equivalent to weak convergence. The following fact which relates convergence in d to convergence in $\wp$ is both easy to prove and extremely useful for establishing weak convergence.

[1] In addition to being important for the present work on percolation, the results of [1] have been applied to convergence of other discrete models to continuous processes. Among those who have subsequently considered the same metric space of curves are: Lawler and Werner [10]; and Kozdron [8].

Proposition 2.2. *Consider the metric space $(\mathcal{S}_\Lambda^*, \mathrm{d})$, and let γ_1, γ_2 be $(\mathcal{S}_\Lambda^*, \mathrm{d})$-valued random variables. If $\mathbf{P}\{\mathrm{d}(\gamma_1, \gamma_2) \geq \varepsilon\} \leq \varepsilon$, then $\wp(\mathcal{L}(\gamma_1), \mathcal{L}(\gamma_2)) \leq \varepsilon$ where $\mathcal{L}(\gamma_i)$ denotes the law of γ_i, $i = 1, 2$.*

Note that in order to simplify notation, Camia and Newman [6] write $\mathcal{S}_\Lambda$ for this metric space $(\mathcal{S}_\Lambda^*, \mathrm{d})$ of curves regarded as equivalence classes of continuous functions from the unit interval to $\mathbb{C}$ modulo monotonic reparametrizations. They also write γ to represent a particular curve and $\gamma(t)$ for a parametrization of γ.

The relevance of this section to the work of Camia and Newman [6] can now be explained. If $\mathbb{P}$ is the configurational measure for i.i.d. Bernoulli(1/2) site percolation on $\mathcal{T}$, then $\mathbb{P}$ induces a probability measure $\mu_{D,a,b}^\delta$ on exploration paths $\gamma_{D,a,b}^\delta$ inside D^δ from a^δ to b^δ. Theorem 1.1 states that as $\delta \downarrow 0$, these measures converge weakly with respect to the metric d given by (1) to $\mu_{D,a,b}$ where $\mu_{D,a,b}$ is the law of $\gamma_{D,a,b}$, the trace of chordal SLE_6 inside D from a to b.

2.2 Riemann mapping theorem

If $D \subset \mathbb{C}$ is a domain, we call a function $f : D \to \mathbb{C}$ *analytic* (or holomorphic) if it is complex differentiable at every point $z \in D$. We say that a function $f : D \to \mathbb{C}$ is *conformal* on D if it is both analytic on D and one-to-one on D (i.e., $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$ for all z_1, $z_2 \in D$). We will sometimes write "let $f : D \to D'$ be a *conformal transformation*." This means that f is conformal on D and onto D' (i.e., $f(D) = D'$). If D and D' are two simply connected domains, we will sometimes write $\mathcal{C}(D, D')$ to denote the set of all conformal transformations from D onto D'. Note that if $f \in \mathcal{C}(D, D')$, then it follows that $f'(z) \neq 0$ for $z \in D$, and $f^{-1} \in \mathcal{C}(D', D)$.

One of the most remarkable, and important, theorems from complex analysis is the Riemann mapping theorem which states that any two simply connected, proper subsets of $\mathbb{C}$ are conformally equivalent. For further details, consult §1.5 of [7].

Theorem 2.3 (Riemann Mapping Theorem)**.** *Suppose that D is a simply connected proper subset of $\mathbb{C}$, and let $z_0 \in D$. Then there exists a unique conformal transformation f of $\mathbb{D}$ onto D satisfying $f(0) = z_0$, $f'(0) > 0$.*

The question of extending the Riemann mapping theorem to the boundary is both an important one and a subtle one. We begin with the following theorem whose proof may be found in [11].

Theorem 2.4 (Continuity Theorem)**.** *If D is a domain whose boundary ∂D is locally connected and $f : \mathbb{D} \to D$ is a conformal transformation, then f can be extended continuously to a map of the closed disk $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$ to $\overline{D}$.*

If ∂D is a Jordan curve, then this continuous extension is also injective as the following theorem due to Carathéodory asserts.

Theorem 2.5 (Carathéodory Extension Theorem)**.** *If D is a domain bounded by a Jordan curve ∂D, and $f : \mathbb{D} \to D$ is a conformal transformation, then f can be extended to a homeomorphism of the closed disk $\overline{\mathbb{D}}$ onto $\overline{D}$.*

2.3 Carathéodory convergence

In many of the examples where a discrete process is proved to converge to a continuous process, the continuous process is described by a family of measures parametrized by (among other things) domains $D \subset \mathbb{C}$. The discrete process is defined on a lattice-type approximation to D in such a way that as the lattice spacing shrinks to 0, the lattice approximation converges to D. The convergence of domains is often in the Carathéodory sense which we now review; see [7] for further details.

Definition 2.6. Fix $r > 0$. Suppose that D_n is a sequence of simply connected Jordan domains with each D_n having the properties that ∂D_n is a finite union of analytic curves and $\mathrm{inrad}(D_n) = r$. The *kernel* of D_n, written $\ker(\{D_n\})$, is the largest domain D containing the origin and having the property that each compact subset of D lies in all but a finite number of the domains D_n. Suppose that $\ker(\{D_n\}) = D$. The sequence D_n *converges in the Carathéodory sense* to D, written $D_n \overset{\text{Cara}}{\to} D$, if every subsequence D_{n_j} of D_n has $\ker(\{D_{n_j}\}) = D$.

Recall that a sequence of functions f_n on a domain D converges to a function f *uniformly on compacta of D* if $f_n \to f$ uniformly on K for each compact $K \subset D$. The following theorem gives an equivalent characterization of Carathéodory convergence which is often easier to apply than the definition. Roughly stated, the convergence of domains in the Carathéodory sense is equivalent to the uniform convergence on compacta of the appropriate Riemann maps. A proof may be found in Theorem 3.1 of [7]. We also let $\mathcal{D}$ denote the set of simply connected Jordan domains containing the origin.

Theorem 2.7 (Carathéodory Convergence)**.** *Suppose that D_n is a sequence of domains with $D_n \in \mathcal{D}$ for each n, and let $f_n \in \mathcal{C}(\mathbb{D}, D_n)$ with $f_n(0) = 0$, $f_n'(0) > 0$. Suppose further that $D \in \mathcal{D}$ and $f \in \mathcal{C}(\mathbb{D}, D)$ with $f(0) = 0$, $f'(0) > 0$. Then $f_n \to f$ uniformly on compacta of $\mathbb{D}$ if and only if $D_n \overset{\text{Cara}}{\to} D$.*

The following two exercises are relatively straightforward applications of Theorem 2.7.

Exercise 2.8. Suppose that $D_n \overset{\text{Cara}}{\to} D$ with D_n, $D \in \mathcal{D}$. Suppose further that there exists an $E \in \mathcal{D}$ with $D_n \subset E$ for all n, and $D \subseteq E$. If $F : E \to \mathbb{D}$ is the conformal transformation with $F(0) = 0$, $F'(0) > 0$, then $F(D_n) \overset{\text{Cara}}{\to} F(D)$.

Exercise 2.9. Suppose that F_n, F are conformal on $\mathbb{D}$. Let $D = F(\mathbb{D})$. If $F_n \to F$ uniformly on compacta of $\mathbb{D}$, then $F_n \circ F^{-1} \to I$ uniformly on compacta of D, where $I : D \to D$ is the identity map $I(z) = z$.

However, it must be noted that the Carathéodory convergence theorem is not enough for the work of Camia and Newman. Instead of delving into these ideas more deeply, we refer the interested reader to Appendix A of [6] for a thorough discussion of the required extensions. The final result of this section is an example to show how one might combine Carathéodory convergence with convergence in the metric d.

Proposition 2.10. *Suppose that F_n, F are conformal mappings of the unit disk $\mathbb{D}$ and that $F_n \to F$ uniformly on compacta of $\mathbb{D}$. If $K \subset \mathbb{D}$ is compact and $\gamma \in \mathcal{S}_K$ with $t_\gamma < \infty$, then $\mathrm{d}(F_n \circ \gamma, F \circ \gamma) \to 0$ as $n \to \infty$.*

We have not yet defined $F \circ \gamma$, but since we are only concerned with curves modulo reparametrization, it doesn't really matter how we do it! One way which keeps track of the parametrization is the following. Let $\gamma : [0, t_\gamma] \to \overline{D}$ and let $F : D \to \mathbb{D}$ be a conformal transformation. Define

$$\alpha_s \equiv \alpha_{s,F,\gamma} \equiv \int_0^s |F'(\gamma(r))|^2 \, \mathrm{d}r \quad \text{and} \quad \sigma_t \equiv \sigma_{t,F,\gamma} \equiv \inf\{s : \alpha_s \geq t\}.$$

If $\alpha_{t_\gamma} < \infty$, and if F extends to the endpoints of γ, then we define $F \circ \gamma : [0, t_{F\circ\gamma}] \to \overline{\mathbb{D}}$, the image of γ under F, by setting $t_{F\circ\gamma} \equiv \alpha_{t_\gamma}$ and $F \circ \gamma(t) \equiv F(\gamma(\sigma_t))$ for $0 \leq t \leq \alpha_{t_\gamma}$ (or equivalently for $0 \leq \sigma_t \leq t_\gamma$). Since $s \mapsto \alpha_{s,F,\gamma}$ is non-negative, continuous, and strictly increasing, it follows that $s \mapsto \sigma_{s,F,\gamma}$ is well-defined.

Proof. Suppose that $\gamma \in \mathcal{S}_K$. Since $F_n \to F$ uniformly on compacta of $\mathbb{D}$, we necessarily have that $F_n \to F$ uniformly on K. Hence, it follows that $t_{F_n\circ\gamma} \to t_{F\circ\gamma}$. Furthermore,

$$\begin{aligned}
&\sup_{0\leq s\leq 1} |F \circ \gamma(t_{F\circ\gamma} s) - F_n \circ \gamma(t_{F_n\circ\gamma} s)| \\
&\quad \leq \sup_{0\leq s\leq 1} |F \circ \gamma(t_{F\circ\gamma} s) - F \circ \gamma(t_{F_n\circ\gamma} s)| + |F \circ \gamma(t_{F_n\circ\gamma} s) - F_n \circ \gamma(t_{F_n\circ\gamma} s)| \to 0
\end{aligned}$$

and the result follows. ∎

3 Cardy's formula

Let D be a bounded simply connected domain containing the origin whose boundary ∂D is a continuous curve. Let $\varphi : \mathbb{D} \to D$ be the unique conformal transformation from the unit disk $\mathbb{D}$ to D with $\varphi(0) = 0$ and $\varphi'(0) > 0$ whose existence is guaranteed by the Riemann mapping theorem. By the continuity theorem, φ has a continuous extension to $\overline{\mathbb{D}}$. (If D is a Jordan domain, then the Carathéodory extension theorem implies this extension is also injective.) Let z_1, z_2, z_3, z_4 be four points of ∂D ordered counterclockwise, and let w_1, w_2, w_3, w_4 be the corresponding four points of $\partial\mathbb{D}$ ordered counterclockwise with $z_j = \varphi(w_j)$; see Figure 4.

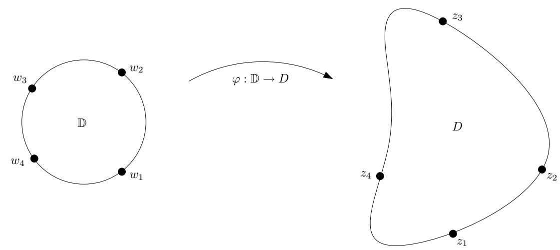

Figure 4: The map $\varphi : \mathbb{D} \to D$ in the setup for Cardy's formula.

Cardy's formula [5] for the probability $\Phi_D(z_1, z_2; z_3, z_4)$ of an "open crossing" in D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is

$$\Phi_D(z_1, z_2; z_3, z_4) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta), \tag{2}$$

where ${}_2F_1$ is a hypergeometric function and

$$\eta = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}$$

denotes the *cross ratio*.

The following observation was first made by L. Carleson. Using properties of the hypergeometric function one can write

$$\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} z^{1/3} {}_2F_1(1/3, 2/3; 4/3; z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

Furthermore, the function

$$z \mapsto \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

is the Schwarz-Christoffel transformation of $\mathbb{H}$ onto the equilateral traingle sending $0 \mapsto 0$, $1 \mapsto 1$, and $\infty \mapsto (1 + i\sqrt{3})/2$. Hence, if D is this equilateral triangle, then Cardy's formula takes the particularly nice form $\Phi_D(1, (1 + i\sqrt{3})/2); 0, x) = x$ as shown in Figure 5.

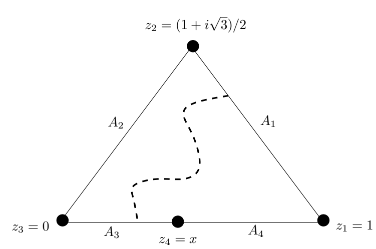

Figure 5: Cardy's formula for an equilateral triangle.

The following important result was announced by Smirnov [13] in 2001. In fact, two lectures by V. Vargas during this current Arbeitsgemeinschaft discussed Smirnov's result in detail. A complete and thorough proof may be found in [3].

Theorem 3.1 (Smirnov [13]). *Let D be a Jordan domain whose boundary ∂D is a finite union of smooth (e.g., C^2) curves. As $\delta \to 0$, the limit of the probability of an open crossing inside D from the counterclockwise arc $\overline{z_1^\delta z_2^\delta}$ to the counterclockwise arc $\overline{z_3^\delta z_4^\delta}$ exists, is a conformal invariant of (D, z_1, z_2, z_3, z_4), and is given by Cardy's formula (2).*

In order to use Smirnov's theorem for the proof of convergence of the exploration path to SLE_6, a slightly extended version was required by Camia and Newman [6]. They proved Theorem 3.2 which extends Smirnov's theorem to a larger class of domains (including all Jordan domains).

We will now briefly discuss the technical matter of the class of domains addressed by Camia and Newman. Suppose that D is a bounded, simply connected domain whose boundary ∂D is a continuous curve. Let a, c, d be three points of ∂D (technically, three prime ends of D) in counterclockwise order. We say that D is *admissible with respect to* (a, c, d) if (i) the counterclockwise arcs $\overline{da}$, $\overline{ac}$, and $\overline{cd}$ are simple curves; (ii) the arc $\overline{cd}$ does not touch the interior of either of the other two arcs; and (iii) from each point in $\overline{cd}$ there is a path to infinity that does not cross ∂D.

Note that if D is Jordan, then D is admissible for any three counterclockwise points $a, c, d \in \partial D$. However, there are domains which arise naturally in the proofs of theorems in [6] that are not Jordan, but are admissible. See Figure 6 for an example.

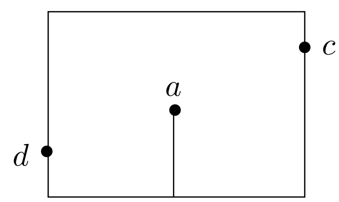

Figure 6: An example of an admissible, non-Jordan domain.

Theorem 3.2 (Camia and Newman [6]). *Consider a sequence $\{(D_k, a_k, c_k, b_k, d_k)\}$ of domains D_k containing the origin, admissible with respect to the points a_k, c_k, d_k on ∂D_k, and with b_k belonging to the interior of the counterclockwise arc $\overline{c_k d_k}$ of ∂D_k. Assume that, as $k \to \infty$, $b_k \to b$ and there is convergence in the metric (1) of the counterclockwise arcs $\overline{d_k a_k}$, $\overline{a_k c_k}$, $\overline{c_k d_k}$ to the corresponding counterclockwise arcs $\overline{da}$, $\overline{ac}$, $\overline{cd}$ of ∂D, where D is a domain containing the origin, admissible with respect to (a, c, d), and b belongs to the interior of $\overline{cd}$. Then, for any sequence $\delta_k \downarrow 0$, the probability $\Phi_{D_k}^{\delta_k}(a_k, c_k; b_k, d_k)$ of an open crossing inside D_k from $\overline{a_k c_k}$ to $\overline{b_k d_k}$ converges as $k \to \infty$ to Cardy's formula $\Phi_D(a, c; b, d)$ for an open crossing inside D from $\overline{ac}$ to $\overline{bd}$ as given by (2).*

4 Characterization of SLE_6

Suppose that D is a simply connected domain whose boundary is a continuous curve, and let a, $b \in \partial D$ be distinct. Suppose further that $\mu_{D,a,b}$ is a probability measure on continuous curves $\gamma = \gamma_{D,a,b} : [0, \infty] \to \overline{D}$ with $\gamma(0) = a$ and $\gamma(\infty) = b$. That is, $\mathcal{L}(\gamma_{D,a,b}) = \mu_{D,a,b}$. Let $D_t = D \setminus K_t$ denote the unique connected component of $D \setminus \gamma[0, t]$ whose closure contains b. We note that this implicitly defines K_t, the *filling of* $\gamma[0, t]$, which is a closed connected subset of $\overline{D}$. We call K_t a *hull* if $\overline{K_t \cap D} = K_t$. Note that if γ is a chordal SLE_κ in $\mathbb{H}$ from 0 to ∞, then $K_t = \gamma[0, t]$ if $0 < \kappa \le 4$, but $K_t \ne \gamma[0, t]$ for $4 < \kappa < 8$ since in this case $\gamma(0, t] \cap \mathbb{R} \ne \emptyset$.

Let $E \subset D$ be a closed subset of $\overline{D}$ such that $a \notin E$, $b \in E$, and $D' = D \setminus E$ is a bounded simply connected domain whose boundary is a continuous curve containing the counterclockwise arc $\overline{cd}$ that does not belong to ∂D (except for its endpoints c and d – see Figure 7).

Let $T = \inf\{t : K_t \cap E \ne \emptyset\}$ be the first time that $\gamma(t)$ hits E. We say that *the hitting distribution of* $\gamma(t)$ *at the stopping time* T *is determined by Cardy's formula* if, for any E and any counterclockwise arc

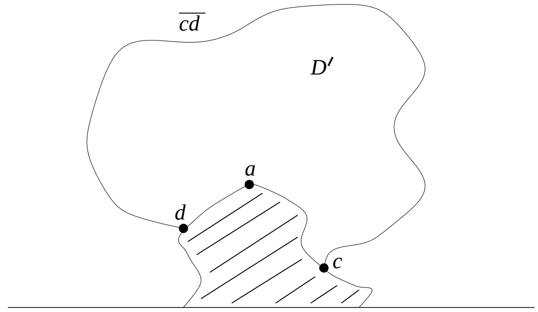

Figure 7: D is the upper half-plane $\mathbb{H}$ with the shaded portion removed, $b = \infty$, E is an unbounded subdomain, and $D' = D \setminus E$ is indicated in the figure. The counterclockwise arc $\overline{cd}$ indicated in the figure belongs to $\partial D'$.

$\overline{xy}$ of $\overline{cd}$, the probability that γ hits E at time T on $\overline{xy}$ is given by

$$\mathbf{P}(\gamma(T) \in \overline{xy}) = \Phi_{D'}(a, c; x, d) - \Phi_{D'}(a, c; y, d). \tag{3}$$

Now let f_0 be a conformal map from the upper half-plane $\mathbb{H}$ to D such that $f_0{}^{-1}(a) = 0$ and $f_0{}^{-1}(b) = \infty$. These two conditions determine f_0 only up to a scaling factor. For $\varepsilon > 0$ fixed, let $C(0, \varepsilon) = \{z : |z| < \varepsilon\} \cap \mathbb{H}$ denote the semi-ball of radius ε centred at 0 on the real line, and let $T_1 = T_1(\varepsilon)$ denote the first time $\gamma(t)$ hits $D \setminus G_1$, where $G_1 \equiv f_0(C(0, \varepsilon))$. Define recursively T_{j+1} as the first time $\gamma[T_j, \infty)$ hits $D_{T_j} \setminus G_{j+1}$, where $D_{T_j} \equiv D \setminus K_{T_j}$, $G_{j+1} \equiv f_{T_j}(C(0, \varepsilon))$, and f_{T_j} is a conformal map from $\mathbb{H}$ to D_{T_j} whose inverse maps $\gamma(T_j)$ to 0 and b to ∞. We choose f_{T_j} so that its inverse is the composition of the restriction of $f_0{}^{-1}$ to D_{T_j} with φ_{T_j} where $\varphi_{T_j} : \mathbb{H} \setminus f_0{}^{-1}(K_{T_j}) \to \mathbb{H}$ is the unique conformal transformation with $\varphi'_{T_j}(\infty) = 1$, and sending $\infty \mapsto \infty$ and $f_0{}^{-1}(\gamma(T_j)) \mapsto 0$. Notice that G_{j+1} is a bounded simply connected domain chosen so that the conformal transformation which maps D_{T_j} to $\mathbb{H}$ maps G_{j+1} to the semi-ball $C(0, \varepsilon)$ centred at the origin on the real line. With these definitions, consider the (discrete-time) stochastic process $X_j \equiv (K_{T_j}, \gamma(T_j))$ for $j = 1, 2, \ldots$. We say that K_t satisfies the *spatial Markov property* if each K_{T_j} is a hull and X_j for $j = 1, 2, \ldots$ is a Markov chain (for any choice of the map f_0).

For example, the conformal invariance and Markovian properties of SLE_6 imply that the hull of chordal SLE_6 satisfies the spatial Markov property.

Theorem 4.1 (Camia and Newman [6]). *If the filling process $\{K_t, t \geq 0\}$ of a continuous curve $\gamma_{D,a,b}$ satisfies the spatial Markov property and its hitting distribution is determined by Cardy's formula, then $\gamma_{D,a,b}$ is distributed like the trace of chordal SLE_6 in D from a to b.*

5 Convergence of the exploration path

The main technical difficulty in the approach of Camia and Newman [6] is to obtain a Markov property for any scaling limit of the percolation exploration path. The difficulty is that in the scaling limit, the exploration path touches itself and the boundary of the domain infinitely often. The standard percolation bound on multiple crossings of a "semi-annulus" only applies to the case of a "flat" boundary. Camia and Newman resolve the issue by proving that Cardy's formula is continuous with respect to changes in the domain. As a final note, we remark that this technical issue is somewhat surprising since an even stronger Markov property trivially holds for the exploration path itself.

Theorem 5.1 (Camia and Newman [6]). *If $\tilde{\gamma}$ is any subsequential limit of the percolation exploration path $\gamma^{\delta}_{D,a,b}$, then $\tilde{K}_t$, the filling process of $\tilde{\gamma}[0, t]$, satisfies the spatial Markov property.*

Acknowledgements

The author would like to express his gratitute to the Mathematisches Forschungsinstitut Oberwolfach (MFO) for hosting the *Arbeitsgemeinschaft: Percolation* from October 8–13, 2007, where this lecture was given. Thanks are owed to V. Beffara, J. van den Berg, and F. Camia who organized the meeting and invited me to present this lecture on the convergence of the percolation exploration path to SLE_6. Special thanks to F. Camia for permission to include Figure 7 from [6], and to P. Nolin for allowing me to include Figure 1.

References

[1] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99:419–453, 1999.

[2] H. Bergström. *Weak Convergence of Measures.* Academic Press, New York, NY, 1982.

[3] B. Bollobás and O. Riordan. *Percolation.* Cambridge University Press, Cambridge, UK, 2006.

[4] V. S. Borkar. *Probability Theory: An Advanced Course.* Springer-Verlag, New York, NY, 1995.

[5] J.L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25:L201–L206, 1992.

[6] F. Camia and C. M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. *Probab. Theory Related Fields*, 139:473–519, 2007.

[7] P. L. Duren. *Univalent Functions*, volume 259 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, New York, NY, 1983.

[8] M. J. Kozdron. On the scaling limit of simple random walk excursion measure in the plane. *ALEA Lat. Am. J. Probab. Math. Stat.*, 2:125–155, 2006.

[9] G. F. Lawler, O. Schramm and W. Werner. One arm exponent for critical 2D percolation. *Electron. J. Probab.*, 7:1–13 (paper no. 2), 2002.

[10] G. F. Lawler and W. Werner. Universality for conformally invariant intersection exponents. *J. Eur. Math. Soc.*, 2:291–328, 2000.

[11] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps*, volume 299 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, New York, NY, 1992.

[12] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[13] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333:239–244, 2001.

[14] S. Smirnov and W. Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8:729–744, 2001.

[15] W. Werner. Lectures on two-dimensional critical percolation. *Lectures given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics.* Preprint, 2007. Available online at `arxiv:math.PR/0710.0856`.