Convergence of 2D critical percolation to SLE_6

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The goal of this lecture is to explain the convergence of the critical site percolation exploration path on the triangular lattice to the trace of chordal SLE_6 for Jordan domains; detailed lecture notes [4] are available from the speaker's website. Our primary reference is the recent paper by Camia and Newman [2] and we cite many results verbatim from that work. The speaker makes no claims of originality, but it is hoped that by highlighting some key elements of the proof in a slightly different way than is done in [2], the interested party can use this present work as a companion to help increase his or her understanding of [2]. At various times in the lecture we will be a little casual with certain hypotheses. We hope that this lack of precision will allow us to better capture the key ideas of [2].

It should be noted that a recent preprint by W. Werner [9] contains lecture notes from a short course given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics. Lecture 3 in those notes is concerned with a proof of this convergence result, but Werner follows a different approach than Camia and Newman. In fact, Werner's notes [9] contain six lectures and a set of exercises on critical site percolation on the triangular lattice that coincide with the topic of this Arbeitsgemeinschaft; we highly recommend reading them.

The primary theorem that we will be concerned with is the following precise formulation of the convergence of the percolation exploration path to SLE_6 as given by the theorem below.

Let $\mathcal{T}$ denote the standard two-dimensional triangular lattice with lattice spacing 1, and let $\mathcal{H}$ denote the hexagonal lattice which is dual to $\mathcal{T}$. For $\delta > 0$, we write $\delta\mathcal{H}$ to denote the hexagonal lattice with lattice spacing δ. Let $D \subset \mathbb{C}$ be a bounded, simply connected Jordan domain. That is, D is a simply connected domain whose boundary ∂D is a Jordan curve (i.e., ∂D is a simple closed curve which is homeomorphic to the unit circle). Suppose that $D^\delta \subset \delta\mathcal{H}$ is a Jordan set which approximates D. That is, D^δ is a simply connected subset of the hexagonal lattice with lattice spacing δ whose external site boundary is a simple closed loop of hexagons such that D^δ is a discrete approximation to D. Suppose further that $a, b \in \partial D$ are distinct boundary points, and let $a^\delta, b^\delta \in \partial D^\delta$ be the corresponding external boundary vertices (or e-vertices). Without being more precise about this exact approximation, we denote by (D, a, b) the simply connected Jordan domain with two distinguished boundary points, and let its δ-scale approximation be denoted by $(D^\delta, a^\delta, b^\delta)$. Essentially, we think of choosing $D^\delta \equiv D \cap \delta\mathcal{H}$. (But this may not produce a simply connected D^δ so we do need to be careful.) A bit more technically, we assume that D^δ, a^δ, and b^δ are chosen so that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ in the Carathéodory sense as $\delta \downarrow 0$. If we now consider D^δ with distinguished e-vertices a^δ and b^δ, then we can see that these two distinguished boundary points partition the (topological) boundary of D^δ into two disjoint arcs. Associate to all external boundary hexagons on one of the arcs the colour "red" and associate to all boundary hexagons on the other arc the colour "white." (The colour red shows up clearly when an electronic version of this note is displayed on screen. However, in this printed version, red appears as grey instead.) Perform critical site percolation on D^δ; that is, for each remaining interior hexagon colour it either red with probability 1/2 or white with probability 1/2. There will be a resulting *interface* joining a^δ with b^δ; that is, a simple path connecting a^δ to b^δ with the property that all hexagons on one side of the path will be white while all hexagons on the other side of the path will be red. We call this path/interface the (critical site) percolation exploration path and denote it by $\gamma^\delta_{D,a,b}$. As $\delta \downarrow 0$, it is this path that converges to chordal SLE_6 in D from a to b.

Figure 1 shows schematically one way of producing the percolation exploration path. Given the realization of the percolation configuration with the boundary conditions (as shown on the left of Figure 1) we then "swallow any islands" by swapping the colour of an "island" with the colour of the "ocean" surrounding it. This produces two disjoint sets—one coloured red and the other coloured white. The percolation exploration path is exactly the interface between these two sets. If we now delete all of the hexagons, then what remains is the percolation exploration path as shown on the right of Figure 1.

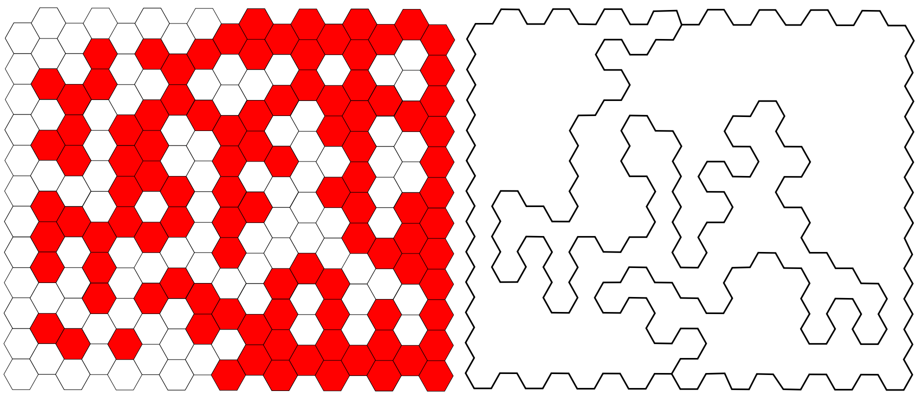

FIGURE 1. On the left is the realization of the percolation configuration with the imposed boundary conditions, and on the right is the resulting exploration path.

Theorem (Camia and Newman [2]). *Let (D, a, b) be a Jordan domain with two distinct selected points on its boundary ∂D, and suppose that $D^\delta \subset \delta\mathcal{H}$ are Jordan sets with two distinct selected e-vertices a^δ, $b^\delta \in \partial D^\delta$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ as $\delta \downarrow 0$. If $\gamma^\delta_{D,a,b}$ denotes the percolation exploration path inside D^δ from a^δ to b^δ, then $\gamma^\delta_{D,a,b}$ converges in distribution as $\delta \downarrow 0$ to $\gamma_{D,a,b}$, the trace of chordal SLE_6 inside D from a to b.*

There are essentially two main parts to the proof. The first is a characterization of SLE_6, and the second is the fact that any subsequential limit of the exploration path satisfies this characterization. The actual proof of the theorem is relatively short *once all of the preliminary lemmas and preparatory theorems have been established*.

Proof. Consider $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ and $\gamma^\delta_{D,a,b}$, the percolation exploration path. The law of $\gamma^\delta_{D,a,b}$ is a distribution on curves. An earlier result of Aizenman and Burchard [1] (in particular, Theorem A.1) is that this family $\gamma^\delta_{D,a,b}$ converges in distribution along subsequential limits $\delta_k \downarrow 0$ to the law of some curve γ. Since the filling of any subsequential limit $\tilde{\gamma}_{D,a,b} \equiv \lim_{\delta_k \downarrow 0} \gamma^{\delta_k}_{D,a,b}$ satisfies the spatial Markov property and the hitting distribution of $\tilde{\gamma}$ is determined by Cardy's formula, it follows that the limit is unique and that the law of $\gamma^\delta_{D,a,b}$ converges as $\delta \downarrow 0$ to the law of $\gamma_{D,a,b}$, the trace of chordal SLE_6 in D from a to b. □

Remark. As a historical note, we mention that a beautiful argument due to Schramm [6] showed that if the scaling limit of the exploration path exists and is conformally invariant, then it must be SLE_κ for some κ. The value $\kappa = 6$ is then obtained by noting that Cardy's formula is satisfied only by SLE_6. The proof of this result was announced by Smirnov in 2001 [7], although a detailed proof did not appear until 2005. The work by Camia and Newman [2] presents that proof in an essentially self-contained form. We also mention that convergence of the exploration path to SLE_6 was used by Smirnov and Werner [8]; and Lawler, Schramm, and Werner [5] to rigorously derive the values of various percolation critical exponents. Camia and Newman also used the convergence to obtain the full scaling limit of critical percolation in two dimensions. Lectures by P. Nolin and C. Hongler during the current Arbeitsgemeinschaft will discuss these critical exponents and the full scaling limit, respectively.

References

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