Excursion Measure in the Plane

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Background and Notation from Complex Analysis

Everything is exclusively two-dimensional. We write w, x, y, z for points in $\mathbb{C}$, and t, n for time ($\in \mathbb{R}$).

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open **unit disk**.

f is **analytic** at $z_0 \in \mathbb{C}$ if it is complex-differentiable at every point in some neighbourhood of z_0.

A single-valued function f is **univalent** in a domain $D \subseteq \mathbb{C}$ if it is one-to-one in D.

An analytic, univalent function is called a **conformal mapping**.

$f : D \to D'$ is a **conformal transformation** if f is a conformal mapping that is onto D'. It follows that $f'(z) \neq 0$ for $z \in D$, and $f^{-1} : D' \to D$ is also a conformal transformation.

Important Results from Complex Analysis

Riemann Mapping Theorem. Suppose D is a simply connected proper subset of $\mathbb{C}$ and $z_0 \in D$. Then there exists a unique conformal transformation f of D onto $\mathbb{D}$ satisfying $f(z_0) = 0$, $f'(z_0) > 0$.

Carathéodory Extension Theorem. Suppose that D is a domain bounded by a Jordan curve ∂D, and let $f : D \to \mathbb{D}$ be a conformal transformation. Then f can be extended to a homeomorphism of $\overline{D} = D \cup \partial D$ onto the closed disk $\overline{\mathbb{D}}$.

Koebe One-Quarter Theorem. If f is a conformal mapping of the unit disk with $f(0) = 0$, then the image of f contains the open disk of radius $|f'(0)|/4$ about the origin.

Subsets of $\mathbb{Z}^2$

Suppose that $A \subset \mathbb{Z}^2$.

Let $\mathcal{A} = \{A \subset \mathbb{Z}^2 : 0 \in A,\ A$ finite and s.c.$\}$.

If $A \in \mathcal{A}$, let $\mathrm{inrad}(A) := \inf\{|z| : z \in \mathbb{Z}^2 \setminus A\}$, $\mathrm{rad}(A) := \sup\{|z| : z \in A\}$, and let

$$\mathcal{A}^n = \{A \in \mathcal{A} : n \leq \mathrm{inrad}(A) \leq 2n\}.$$

There are 3 reasonable ways to define the **boundary** of A.

- **(outer) boundary:** $\partial A := \{y \in \mathbb{Z}^2 \setminus A : |y - x| = 1$ for some $x \in A\}$

- **inner boundary:**
  $\partial_i A := \partial(\mathbb{Z}^2 \setminus A) = \{x \in A : |y - x| = 1$ for some $y \in \mathbb{Z}^2 \setminus A\}$

- **edge boundary:** $\partial_e A := \{(x, y) : x \in A,\ y \in \mathbb{Z}^2 \setminus A,\ |x - y| = 1\}$

Subsets of $\mathbb{C}$

A **domain** $D \subset \mathbb{C}$ is an open subset of $\mathbb{C}$.

Define $\operatorname{inrad}(D) = \inf\{|z| : z \in \mathbb{C} \setminus D\}$ and $\operatorname{rad}(D) := \sup\{|z| : z \in \partial D\}$.

$\mathcal{D} := \{D \subset \mathbb{C} : 0 \in D;\ D$ s.c., bounded; ∂D Jordan, piecewise analytic$\}$

For D, $D' \in \mathcal{D}$, let $\mathcal{T}(D, D') := \{$conformal transformations $f : D \to D'\}$.

By the Riemann mapping theorem, $\mathcal{T}(D, D') \neq \emptyset$.

By the Carathéodory extension theorem, if $f \in \mathcal{T}(D, D')$, then f can be extended to a homeomorphism of $\overline{D}$ onto $\overline{D'}$.

$\tilde{A} \subset \mathbb{C}$ *Associated to* $A \subset \mathbb{Z}^2$

We associate a domain $\tilde{A} \subset \mathbb{C}$ to each finite $A \subset \mathbb{Z}^2$.

Put

$$\tilde{A} \cup \partial \tilde{A} = \bigcup_{x \in A} \mathcal{S}_x ,$$

where $\mathcal{S}_x$ is the closed square of side one centred at x whose sides are parallel to the coordinate axes.

Let $\tilde{A}$ denote the open subset of $\mathbb{C}$ bounded by $\partial \tilde{A}$ containing A.

Note. $\tilde{A}$ is s.c. domain iff A is s.c. subset of $\mathbb{Z}^2$.

Note. If $A \in \mathcal{A}$, then $\tilde{A} \in \mathcal{D}$.

Carathéodory Convergence

The notion of convergence of domains in $\mathbb{C}$ in the Carathéodory sense is different than the usual topological convergence of domains.

Let domains $E_n, E \subset \mathbb{C}$.

Let

- $f_n \in \mathcal{T}(\mathbb{D}, E_n)$ with $f_n(0) = 0$, $f_n'(0) > 0$,
- $f \in \mathcal{T}(\mathbb{D}, E)$ and $f(0) = 0$, $f'(0) > 0$.

Definition and Theorem. E_n converges to E in the **Carathéodory sense** if $f_n \to f$ uniformly on every compact subsets of $\mathbb{D}$.

Let $D \subset \mathbb{C}$ be simply connected with $0 \in D$, $\mathrm{inrad}(D) = 1$, and $\mathrm{rad}(D) = R$.

Let $D'_N = \{x \in \frac{1}{N}\mathbb{Z}^2 \cap D : \frac{1}{N}\mathcal{S}_x \subseteq D\}$.

Let D_N be connected component of D'_N containing the origin.

Let $\tilde{D}_N$ be the union of scaled squares so that

$$\tilde{D}_N \cup \partial \tilde{D}_N = \bigcup_{x \in D_N} \frac{1}{N}\mathcal{S}_x .$$

Note. $\tilde{D}_N \in \mathcal{D}$.

Theorem.

$$\tilde{D}_N \overset{\text{Cara}}{\to} D$$

Background from Probability

A **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space with total measure $\mathbb{P}(\Omega) = 1$.

A **random variable** is a measurable mapping $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{C}, \mathcal{B})$.

X induces a probability measure on $(\mathbb{C}, \mathcal{B})$ called the **law** of Y, $\mathcal{L}_Y := \mathbb{P} \circ Y^{-1}$, defined as $\mathcal{L}_Y(A) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in A\})$ for each $A \in \mathcal{B}$.

A **stochastic process** is a collection of random variables $\{Y_i : i \in I\}$ for some indexing set I.

Simple Random Walk

Let X_i be i.i.d. with $\mathbb{P}\{X_i = e\} = 1/4$, $|e| = 1$, and set

$$S_n = x + X_1 + \cdots + X_n.$$

The process $\{S_n : n \in \mathbb{N}\}$ is a **simple random walk** on $\mathbb{Z}^2$ starting at $x \in \mathbb{Z}^2$.

Write $S[0, j] = [S(0), S(1), \ldots, S(j)]$ for the set of points visited by the SRW (in order).

Write $\mathbb{P}^x$ for the probability starting at x.

Complex Brownian Motion

The process $\{B_t, t \geq 0\}$ is a complex Brownian motion (starting at $x \in \mathbb{C}$) if

- $\mathbb{P}(B_0 = x) = 1$ and the function $t \mapsto B_t$ is continuous (wp1),
- for any $t_0 < t_1 < \ldots < t_n$ the increments $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_{n-1}} - B_{t_n}$ are independent,
- for any s, $t \geq 0$, the increment $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ is normally distributed.

Write $B[0,T] = \{z \in \mathbb{C} : B_s = z \text{ some } 0 \leq s \leq T\}$.

Write $\mathbb{P}^x$ for the probability starting at x.

Exit Times

Suppose that $A \subset \mathbb{Z}^2$.

Let $\tau_A = \inf\{n : S_n \notin A\} = \inf\{n : S_n \in \partial A\}$.

We call τ_A the **exit time** of random walk from A (or the hitting time of A^c).

τ_A is a **stopping time** for any A.

Suppose that $D \subset \mathbb{C}$.

Let $T_D = \inf\{t : B_t \notin D\}$.

We call T_D the **exit time** of Brownian motion from D (or the hitting time of D^c).

T_D is a **stopping time** provided $D \in \mathcal{B}$.

Discrete Hitting Measure

Suppose that $x \in A$. For $y \in \mathbb{Z}^2$, let

$$H_A(x,y) = \mathbb{P}^x\{S(\tau_A) = y\}$$

be the hitting probability of y from x.

Since $H_A(x,y) = 0$ if $y \notin \partial A$, $H_A(x,\cdot)$ is a probability measure on $\mathbb{Z}^2$ concentrated on ∂A.

$H_A(x,y)$ is the discrete analogue of the Poisson kernel.

If $V \subseteq \partial A$, then $\mathbb{P}^x\{S(\tau_A) \in V\} = \sum_{y \in V} H_A(x,y)$.

Poisson Kernel

Suppose $D \in \mathcal{D}$.

Write $\mathbb{P}^z\{B_{T_D} \in \mathrm{d}y\}$ for **harmonic measure** in D from $z \in D$.

Its density (Radon-Nikodým derivative) with respect to arclength is $H_D(z, y)$, the **Poisson kernel**.

i.e., $\mathbb{P}^z\{B_{T_D} \in \mathrm{d}y\} = H_D(z, y)\,|\mathrm{d}y|$, where $|\mathrm{d}y|$ is arclength measure on ∂D.

If $V \subseteq \partial D$, then $\mathbb{P}^z\{B(T_D) \in V\} = \displaystyle\int_V H_D(z, y)\,|\mathrm{d}y|$.

Ex. $H_{\mathbb{D}}(z, y) = \dfrac{1}{2\pi}\,\dfrac{1 - |z|^2}{|y - z|^2}$ for $z \in \mathbb{D}$, $|y| = 1$.

Wiener Measure

Write $\mu_D(z)$ for Wiener measure, the measure on curves starting at z, ending at ∂D.

It is well-known that Wiener measure is the law of BM $\{B_t, 0 \le t \le T_D\}$.

Note. There is a question of the measure space of curves $\mathcal{K}$, and the appropriate σ-algebra, $\mathcal{B}(\mathcal{K})$. We do not go into those details here. However, when we write measures of the form $\mu_D(z)$, we really mean that there are two parameters associated with this measure, D and z, and that $\mu_D(z) := \mu_{D,z}(\cdot)$ for $\cdot \in \mathcal{B}(\mathcal{K})$.

We can write $\mu_D(z) = \int_{\partial D} \mu_D(z,y)\,|\mathrm{d}y|$ where $\mu_D(z,y)$ is the measure on curves starting at $z \in D$, ending at $y \in \partial D$.

$\mu_D(z,y)$ is a finite measure with mass $H_D(z,y)$.

The probability measure

$$\mu_D^{\#}(z,y) = \frac{\mu_D(z,y)}{H_D(z,y)}$$

is the law of BM starting at z conditioned to exit D at y.

Conformal Invariance

Paul Lévy first showed that BM is conformally invariant.

Let D, $D' \in \mathcal{D}$ and let $f \in \mathcal{T}(D, D')$.

Then

- $\mathbb{P}^z\{B_{T_D} \in \mathrm{d}y\} = \mathbb{P}^{f(z)}\{B'_{T_{D'}} \in f(\mathrm{d}y)\}$

- $f \circ \mu_D(z) = \mu_{D'}(f(z))$

- $H_D(z, y) = |f'(y)|\, H_{D'}(f(z), f(y))$

- $f \circ \mu_D(z, y) = |f'(y)|\, \mu_{D'}(f(z), f(y))$

where B' is another Brownian motion.

i.e., the conformal image of Brownian motion in D is a time-change of another Brownian motion stopped on exiting D'.

Excursions

An **excursion** in D is a curve $\gamma : [0, t_\gamma] \to \mathbb{C}$ with

- $0 < t_\gamma < \infty$,
- $\gamma(0) \in \partial D$,
- $\gamma(t_\gamma) \in \partial D$, and
- $\gamma(0, t_\gamma) \subset D$.

If $\gamma(0) = x$ and $\gamma(t_\gamma) = y$, then γ is called an **excursion from** x **to** y **in** D.

Excursion Poisson Kernel

For x, $y \in \partial D$, the **excursion Poisson kernel** is

$$H_{\partial D}(x,y) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y)$$

where $\mathbf{n}_x$ is the (inward pointing) unit normal vector to D at x.

i.e., the excursion Poisson kernel is the normal derivative of the (analytically continued) Poisson kernel.

Ex. If $x = e^{i\theta}$, $y = e^{i\theta'} \in \partial\mathbb{D}$, $y \neq x$, then

$$H_{\partial\mathbb{D}}(x,y) = \frac{1}{\pi} \frac{1}{|y-x|^2} = \frac{1}{2\pi} \frac{1}{1 - \cos(\theta' - \theta)}.$$

Excursion Measure

Let $\mu_{\partial D}(x, y)$ be the measure on excursions from x to y in D.

$\mu_{\partial D}(x, y)$ is a finite measure with mass $H_{\partial D}(x, y)$.

Think of $H_{\partial D}(x, y)$ as " $\mathbb{P}^x\{$excursion γ ends at $y\}$ ".

Consider the measure

$$\mu^{\#}_{\partial D}(x, y) := \frac{\mu_{\partial D}(x, y)}{H_{\partial D}(x, y)},$$

excursion measure normalized to be a probability measure.

Proposition.

$$\mu_{\partial D}(x, y) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mu_D(x + \varepsilon \mathbf{n}_x, y)$$

Question of existence?

Conditioned BM. If $z \in D$, then using Doob's h-processes it is easy to construct Brownian motion conditioned to exit D at y.

Furthermore, the probability measure

$$\lim_{\varepsilon \downarrow 0} \mu_D(x + \varepsilon \mathbf{n}_x, y) =: \mu^{\#}_{\partial D}(x, y)$$

exists (in the Prohorov metric).

Thus, $\mu_{\partial D}(x, y) = \mu^{\#}_{\partial D}(x, y) \cdot H_{\partial D}(x, y)$ exists.

σ-finite Excursion Measures

Let

$$\mu_{\partial D}(x) = \int_{\partial D} \mu_{\partial D}(x, y)\, |\mathrm{d}y|$$

be excursion measure on excursion from x in D.

Note that $\mu_{\partial D}(x)$ is an infinite, but σ-finite, measure.

Let

$$\mu_{\partial D} = \int_{\partial D} \mu_{\partial D}(x)\, |\mathrm{d}x| = \int_{\partial D} \int_{\partial D} \mu_{\partial D}(x, y)\, |\mathrm{d}x|\, |\mathrm{d}y|$$

be excursion measure on excursion in D.

Note that $\mu_{\partial D}$ is also an infinite, but σ-finite, measure.

Conformal Invariance

Let D, $D' \in \mathcal{D}$, let $f \in \mathcal{T}(D, D')$, and suppose that ∂D, $\partial D'$ are locally analytic at x, y, and $f(x)$, $f(y)$, resp.

Then

- $H_{\partial D}(x, y) = |f'(x)|\,|f'(y)|\,H_{\partial D'}(f(x), f(y))$
- $f \circ \mu_{\partial D}(x, y) = |f'(x)|\,|f'(y)|\,\mu_{\partial D'}(f(x), f(y))$
- $f \circ \mu_{\partial D}(x) = |f'(x)|\,\mu_{\partial D'}(f(x))$
- $f \circ \mu_{\partial D} = \mu_{\partial D'}$

Green's Functions for $\mathbb{C}$

For x, $y \in \mathbb{D}$, let

$$g_{\mathbb{D}}(x,y) = \log \left| \frac{\overline{y}x - 1}{y - x} \right|$$

denote the standard Green's function in $\mathbb{D}$.

If $x \in D$, we can define $g_D(x,\cdot)$ as the unique harmonic function on $D \setminus \{x\}$, vanishing on ∂D, with $g_D(x,y) = -\log|x-y| + O(1)$ as $|x-y| \to 0$.

This implies for x, $y \in D$, $x \neq y$,

$$g_D(x,y) = \mathbb{E}^x[\log|B(T_D) - y|] - \log|x-y|.$$

Note that $g_D(x,y) = g_D(y,x)$ and $g_D(x) := g_D(0,x) = g_D(x,0)$.

If $D \in \mathcal{D}$, $f \in \mathcal{T}(D,\mathbb{D})$ with $f(0) = 0$, $f'(0) > 0$, then the Green's function for D is

$$g_D(x,y) = g_{\mathbb{D}}(f(x), f(y))$$

for x, $y \in D$.

Green's Functions for $\mathbb{Z}^2$

Suppose $A \subset \mathbb{Z}^2$. For x, $y \in A$, let

$$G_A(x,y) := \mathbb{E}^x[\sum_{j=0}^{\tau_A - 1} \mathbb{1}\{S_j = y\}] = \sum_{j=0}^{\infty} \mathbb{P}^x\{S_j = y, \tau_A > j\}$$

denote the Green's function (for simple random walk) on A.

This is the expected number of visits to y starting at x before exiting A.

Note that $G_A(x,y) = G_A(y,x)$ and $G_A(x) := G_A(x,0) = G_A(0,x)$.

Fact. For $x \in A$,

$$G_A(x) = \mathbb{E}^x[a(S(\tau_A))] - a(x)$$

where a is the potential kernel for simple random walk defined by

$$a(x) := \lim_{m \to \infty} \sum_{j=0}^{m} [\mathbb{P}^0\{S_j = 0\} - \mathbb{P}^x\{S_j = 0\}].$$

Fact. $a(x) = \frac{2}{\pi} \log|x| + k_0 + o(|x|^{-3/2})$ as $|x| \to \infty$, where $k_0 = (2\varsigma + 3\ln 2)/\pi$ and ς is Euler's constant.

Continuous and Discrete Beurling Estimates

Generalized Beurling Projection Theorem. There is a constant $c < \infty$ such that if $\gamma : [0,1] \to \mathbb{C}$ is a curve with $\gamma(0) = 0$, $|\gamma(1)| = 1$, $\gamma(0,1) \subset \mathbb{D}$, and $x \in \mathbb{D}$, then

$$\mathbb{P}^x\{B[0, T_{\mathbb{D}}] \cap \gamma[0,1] = \emptyset\} \leq c\,|x|^{1/2}.$$

Beurling Estimate. There is a constant $c < \infty$ such that if $x \in \tilde{A}$, then for all $r > 0$,

$$\mathbb{P}^x\{|B(T_A) - x| > r \ \mathrm{dist}(x, \partial\tilde{A})\} \leq c\, r^{-1/2}.$$

Discrete Beurling Projection Theorem. If $A \in \mathcal{A}$ is of radius R, then

$$\lim_{|x|\to\infty} \mathbb{P}^x\{S(\tau_{A^c}) = 0\} = O(R^{-1/2}).$$

Discrete Beurling Estimate There is a constant $c < \infty$ such that if $x \in A$, then for all $r > 0$,

$$\mathbb{P}^x\{|S(\tau_A) - x| > r \ \mathrm{dist}(x, \partial A)\} \leq c\, r^{-1/2}.$$

Consequences of the Continuous Beurling Estimate

Suppose that $A \in \mathcal{A}^n$ with associated domain $\tilde{A} \subset \mathbb{C}$.

Let $f_A = f_{\tilde{A}} \in \mathcal{T}(\tilde{A}, \mathbb{D})$ with $f_A(0) = 0$, $f'_A(0) > 0$.

Let $g_A = g_{\tilde{A}}$ be the Green's function.

Fact. $f_A(x) = \exp\{-g_A(x) + i\theta_A(x)\}$

If $|x| \geq n/4$, then

$$g_A(x) \leq c\, n^{-1/2} \ \mathrm{dist}(x, \partial\tilde{A})^{1/2}.$$

If $x \in \partial_i A$, then $g_A(x) \leq cn^{-1/2}$; hence

$$f_A(x) = \exp\{i\theta_A(x)\} + O(n^{-1/2}).$$

If $z \in \partial A$, let $\theta_A(z)$ be the average of $\theta_A(x)$ over all $x \in A$ with $|x - z| = 1$.

The Beurling estimate and a simple Harnack principle show that if $(x, z) \in \partial_e A$, then

$$\theta_A(z) = \theta_A(x) + O(n^{-1/2}).$$

Consequences of the Discrete Beurling Estimate

If $A \in \mathcal{A}^n$ and $|x| \geq n/4$, then

$$G_A(x) \leq c\, n^{-1/2} \ \mathrm{dist}(x, \partial A)^{1/2}.$$

If $x \in \partial_i A$, then $G_A(x) \leq cn^{-1/2}$.

Fact. If $0 \neq x \in \partial_i A$, then

$$G_A(0) = G_{A\setminus\{x\}}(0) + \frac{G_A(x)^2}{G_A(x,x)}.$$

Thus, if we replace $A \setminus \{x\}$ in the above formula with the connected component of $A \setminus \{x\}$ containing the origin, then

$$G_A(0) - G_{A\setminus\{x\}}(0) \leq G_A(x)^2 \leq c\, n^{-1}.$$

Relationship between g_A and G_A

Proposition. There exists a constant c such that for every n, B_t and S_t can be defined on the same probability space so that if $A \in \mathcal{A}^n$, $1 < r \le n^{20}$, and $x \in A$ with $|x| \le n^3$,

$$\mathbb{P}^x\{|B(T_A) - S(\tau_A)| \ge c\, r\ \log n\} \le c\, r^{-1/2}.$$

Corollary. There exists a c such that if $A \in \mathcal{A}^n$ and $|x| \le n^2$,

$$|\ \mathbb{E}^x[\log |B(T_A)|] - \mathbb{E}^x[\log |S(\tau_A)|]\ | \le c\, n^{-1/3}\ \log n.$$

Corollary. If $A \in \mathcal{A}^n$, then

$$G_A(x) = (2/\pi)\, g_A(x) + k_x + O(n^{-1/3} \log n),$$

$k_x = k_0 + (2/\pi)\ \log |x| - a(x)$. Note $|k_x| \le c|x|^{-3/2}$.

For any $A \in \mathcal{A}^n$, let $A^{*,n} = \{x \in A : g_A(x) \ge n^{-1/16}\}$.

Corollary. If $A \in \mathcal{A}^n$, and $x \in A^{*,n}, y \in A$, then

$$G_A(x,y) = (2/\pi)\, g_A(x,y) + k_{y-x} + O(n^{-7/24} \log n).$$

Green's Function Estimates Away from the Boundary

Let D_N be the $1/N$ scale discrete approximation to D and set $2ND_N := A_N \in \mathcal{A}^n$ with associated domain $\widetilde{(2ND_N)} = \tilde{A}_N$.

Recall that for $A_N \in \mathcal{A}^N$,

$$A_N^* = \{x \in A_N : g_{A_N}(x) \geq N^{-1/16}\}.$$

Let $x \in D_N$ be such that $2Nx \in (2ND_N)^* = A_N^*$.

Let $y \in D_N$ with $2Ny \in A_N$ and $|x - y| \geq N^{-29/36}$.

Then,

$$G_{D_N}(x, y) = \frac{2}{\pi} g_{D_N}(x, y) + O(N^{-7/24} \log N).$$

An Estimate for Hitting the Boundary

If BM has a good chance of exiting $\tilde{A}$ at some subset of $\tilde{V} \subseteq \partial\tilde{A}$, then there is also a good chance that SRW exits A at the corresponding set $V \subseteq \partial A$.

Let $V \subseteq \partial A$ with associated set $\tilde{V} \subseteq \partial\tilde{A}$.

Let $\mathbb{P}^x\{S(\tau_A) \in V\} = \sum_{y \in V} H_A(x, y) =: h(x)$.

Let $\mathbb{P}^z\{B(T_D) \in V\} = \int_V H_D(z, y)\,|\mathrm{d}y| =: \tilde{h}(x)$.

Proposition. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $A \in \mathcal{A}^n$, $V \subseteq \partial A$, and $x \in A$ with $h(x) > \varepsilon$, then $\tilde{h}(x) > \delta$.

Green's Function Estimates

Fact. If $z = f_A(x) = (1-r)e^{i\theta}$, $z' = f_A(y) \in \mathbb{D}$ with $|z - z'| \geq r$, then

$$g_A(x,y) = g_{\mathbb{D}}(z,z') = \frac{g_{\mathbb{D}}(z)\ (1-|z'|^2)}{|z' - e^{i\theta}|^2}\ [1 + O(\frac{r}{|z-z'|})].$$

Similarly, if $z' = f_A(y) = (1-r')e^{i\theta'}$ with $r \geq r'$ and $|z - z'| \geq r$,

$$g_A(x,y) = g_{\mathbb{D}}(z,z') = \frac{g_{\mathbb{D}}(z)\ g_{\mathbb{D}}(z')}{1-\cos(\theta-\theta')}\ [1 + O(\frac{r}{|\theta-\theta'|})].$$

Let $J_{x,n} = \{z \in A : |f_A(z) - \exp\{i\theta_A(x)\}| \geq n^{-1/16}\log^2 n\}$.

Proposition. Suppose $A \in \mathcal{A}^n$ and $x \in A \setminus A^{*,n}$, $y \in J_{x,n}$. Then,

$$G_A(x,y) = G_A(x)\ \frac{1-|f_A(y)|^2}{|f_A(y) - e^{i\theta_A(x)}|^2}\ [\,1 + O(\frac{n^{-1/16}\log n}{|f_A(y) - e^{i\theta_A(x)}|})\,], \quad y \in A^{*,n},$$

$$G_A(x,y) = \frac{(\pi/2)\ G_A(x)\ G_A(y)}{1-\cos(\theta_A(x) - \theta_A(y))}\ [\,1 + O(\frac{n^{-1/16}\log n}{|\theta_A(y) - \theta_A(x)|})\,], \quad y \in A \setminus A^{*,n}.$$

Hitting Probability Estimates

If $A \in \mathcal{A}^n$, $x \in \partial_i A$, $y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then

$$H_A(x,y) = \frac{(\pi/2)\ G_A(x)\ H_A(0,y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\ [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})].$$

Similarly, if $x \in \partial A$, $y \in \partial A$, let $H_A(x,y)$ be the probability that a simple random walk starting at x takes its first step into A and then exits A at y.

If $A \in \mathcal{A}^n$, x, $y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then

$$H_A(x,y) = \frac{(\pi/2)\ H_A(0,x)\ H_A(0,y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\ [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})].$$

Loop-erased Random Walk

Suppose $S = [S(0), S(1), \ldots, S(m)]$ is a simple random walk path of length m

Let $\Lambda(S)$ be the loop-erased part of S, which can be constructed as follows.

If S is already self-avoiding, set $\Lambda(S) = S$. Otherwise, let $s_0 = \max\{j : S(j) = S(0)\}$.

For $i > 0$, let $s_i = \max\{j : S(j) = S(s_{j-1} + 1)\}$.

If we let $n = \min\{i : s_i = m\}$, then $\Lambda(S) = [S(s_0), S(s_1), \ldots, S(s_n)]$.

Although simple random walk in $\mathbb{Z}^2$ is recurrent, the loop-erasing procedure makes perfect sense for any finite segment.

In particular, since $\tau_A < \infty$ a.s., there is no problem discussing $\Lambda([S(0), S(1), \ldots, S(\tau_A)])$.

Note. It is not possible to define loop-erased Brownian motion in an analogous way since loops exist at arbitrarily small scales.

Fomin's Identity, I

Suppose that $A \in \mathcal{A}^n$ and $x^1, x^2, \ldots, x^N \in \partial A$.

Let $S^1, S^2, \ldots, S^N$ be independent simple random walks starting at $x^1, x^2, \ldots, x^N$, respectively.

Set $\tau_A^k := \inf\{j > 0 : S_j^k \notin A\}$.

Let $L^k = \Lambda(S^k)$ be the loop erasure of the path $[S^k(0) = x^k, S^k(1), \ldots, S^k(\tau_A^k)]$.

Let $\mathcal{E} = \mathcal{E}(x^1, \ldots, x^N, y^1, \ldots, y^N; A)$ be the event

- $S^k(\tau_A^k) = y^k, \quad k = 1, \ldots, N$, and
- $S^k[0, \tau_A^k] \cap \{L^1 \cup \cdots \cup L^{k-1}\} = \emptyset, \quad k = 2, \ldots, N.$

Fomin's Identity, II

Theorem (Fomin).

$$\mathbb{P}\{\mathcal{E}\} = \det[\mathbf{H}_A],$$

where $\mathbf{H}_A = [H_A(x^k, y^\ell)]$ is the $N \times N$ hitting matrix

$$\mathbf{H}_A = \begin{bmatrix} H_A(x^1, y^1) & \cdots & H_A(x^1, y^N) \\ \vdots & \ddots & \vdots \\ H_A(x^N, y^1) & \cdots & H_A(x^N, y^N) \end{bmatrix}$$

Consequences

Theorem. Suppose that $A \in \mathcal{A}^n$ and $x^1, \ldots, x^N, y^1, \ldots, y^N \in \partial A$ with

$$\delta = \min_{1 \leq k, \ell \leq N} \{|\theta_A(x^k) - \theta_A(y^\ell)|\} \geq \varepsilon_n.$$

Let $\varphi_A(x^k, y^\ell) = [1 - \cos(\theta_A(x^k) - \theta_A(y^\ell))]^{-1}$. If $\mathcal{E}$ is the event defined as before, then

$$\mathbb{P}\{\mathcal{E}\} = (\pi/2)^N \; [\prod_{k=1}^N H_A(0, x^k)] \; [\prod_{\ell=1}^N H_A(0, y^\ell)] \; \det[\mathbf{\Phi}_A] \; [1 + O(\varepsilon_n^3 \delta^{-1})]$$

where $\mathbf{\Phi}_A$ is the $N \times N$ matrix $\mathbf{\Phi}_A = [\varphi_A(x^k, y^\ell)]$.

Note.

$$\varphi_A(x^k, y^\ell) = 2\pi \, H_{\partial \mathbb{D}}(e^{\theta_A(x^k)}, e^{\theta_A(y^\ell)})$$

Scaling Limit

Suppose that $D \subseteq \mathbb{C}$ is a simply connected domain; ∂_1 and ∂_2 are disjoint non-trivial subarcs of ∂D; D_N is the N-scale approximate to D; $\tilde{D_N}$ is the associated domain; $\partial_{\tilde{N},1}$ and $\partial_{\tilde{N},2}$ are the associated subarcs.

Then, as $N \to \infty$,

$$\sum_{x^1,\ldots,x^k \in \partial_1^N} \sum_{y^1,\ldots,y^k \in \partial_2^N} \det[H_{\partial D^N}(x^j, y^{j'})]_{1 \leq j,j' \leq k}$$

converges to a conformally invariant limit. In fact, this limit is

$$\int_{(\partial_1)^k} \int_{(\partial_2)^k} \det[H_{\partial D}(x^j, y^{j'})]_{1 \leq j,j' \leq k} \, |\mathrm{d}x^1| \cdots |\mathrm{d}x^k| \, |\mathrm{d}y^1| \cdots |\mathrm{d}y^k|.$$

Furthermore, the measure on simple random walk excursions $\mu^{\mathsf{RW}}_{D,N}(\partial_{N,1}, \partial_{N,2})$ coverges to the measure on excursions $\mu_{\partial D_N}(\partial_{\tilde{N},1}, \partial_{\tilde{N},2})$.

And, the measure on excursions $\mu_{\partial D_N}(\partial_{\tilde{N},1}, \partial_{\tilde{N},2})$ converges to $\mu_{\partial D}(\partial_1, \partial_2)$.