Multiple SLE paths and Fomin's identity

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This talk is based on joint work with **Greg Lawler** of the University of Chicago.

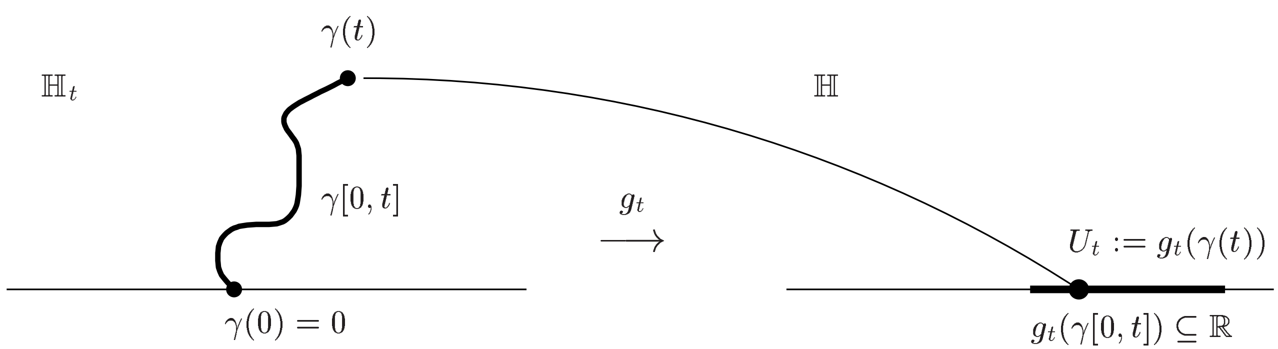
Review of SLE

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane, and consider a simple (non-self-intersecting) curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$.

For every fixed $t \geq 0$, the slit plane $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0, t]$ is simply connected and so by the Riemann mapping theorem, there exists a unique conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$ which can be expanded as

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty,$$

where $b(t) = \mathrm{hcap}(\gamma(0, t])$ is the **half-plane capacity** of γ up to time t.



It can be shown that there is a unique point $U_t \in \mathbb{R}$ for all $t \geq 0$ with $U_t := g_t(\gamma(t))$ and that the function $t \mapsto U_t$ is continuous.

Review of SLE (cont)

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty, \quad \mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t]$$

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by a PDE involving U_t.

This is due to C. Loewner (1923) who showed that if γ is a curve as above such that its half-plane capacity $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$, then for $z \in \mathbb{H}$ with $z \notin \gamma[0,\infty)$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the PDE

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Note that if $b(t) \in C^1$ is an increasing function, then we can reparametrize the curve γ so that $\mathrm{hcap}(\gamma(0,t]) = b(t)$. This is the so-called **parametrization by capacity**.

Review of SLE (cont)

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{$*$}$$

The obvious thing to do now is to start with a continuous function $t \mapsto U_t$ from $[0, \infty)$ to $\mathbb{R}$ and solve the Loewner equation for g_t.

Ideally, we would like to solve ($*$) for g_t, define simple curves $\gamma(t)$, $t \geq 0$, by setting $\gamma(t) = g_t^{-1}(U_t)$, and have g_t map $\mathbb{H} \setminus \gamma(0, t]$ conformally onto $\mathbb{H}$.

Although this is the intuition, it is not quite precise because we see from the denominator on the right-side of ($*$) that problems can occur if $g_t(z) - U_t = 0$.

Formally, if we let T_z be the supremum of all t such that the solution to ($*$) is well-defined up to time t with $g_t(z) \in \mathbb{H}$, and we define $\mathbb{H}_t = \{z : T_z > t\}$, then g_t is the unique conformal transformation of $\mathbb{H}_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $t \to \infty$.

Review of SLE (cont)

The novel idea of Schramm was to take the continuous function U_t to be a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

The **chordal Schramm-Loewner evolution with parameter** $\kappa \geq 0$ **with the standard parametrization** (or simply SLE_κ) is the random collection of conformal maps $\{g_t, \, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}\, W_t}, \quad g_0(z) = z, \tag{LE}$$

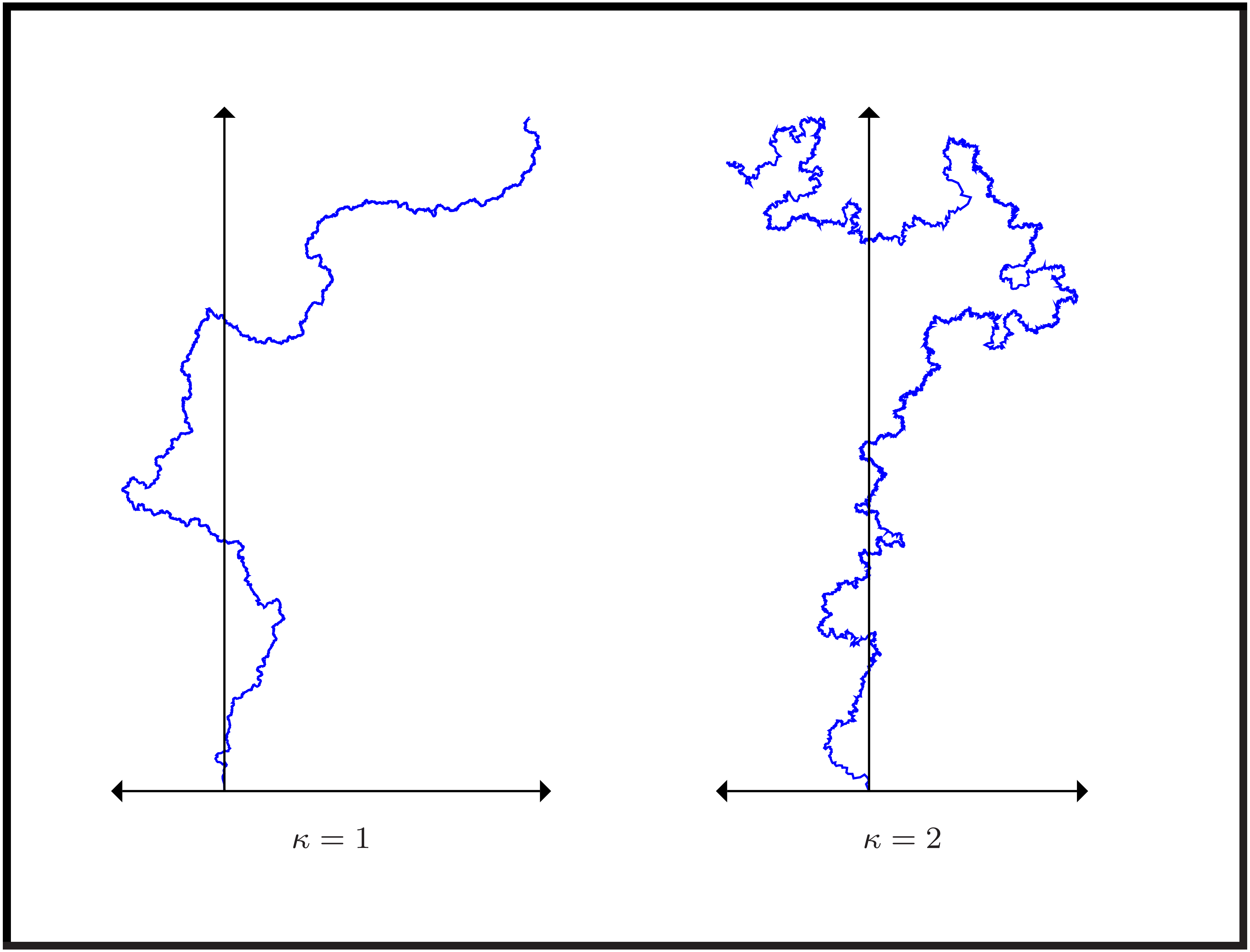
where W_t is a standard one-dimensional Brownian motion.

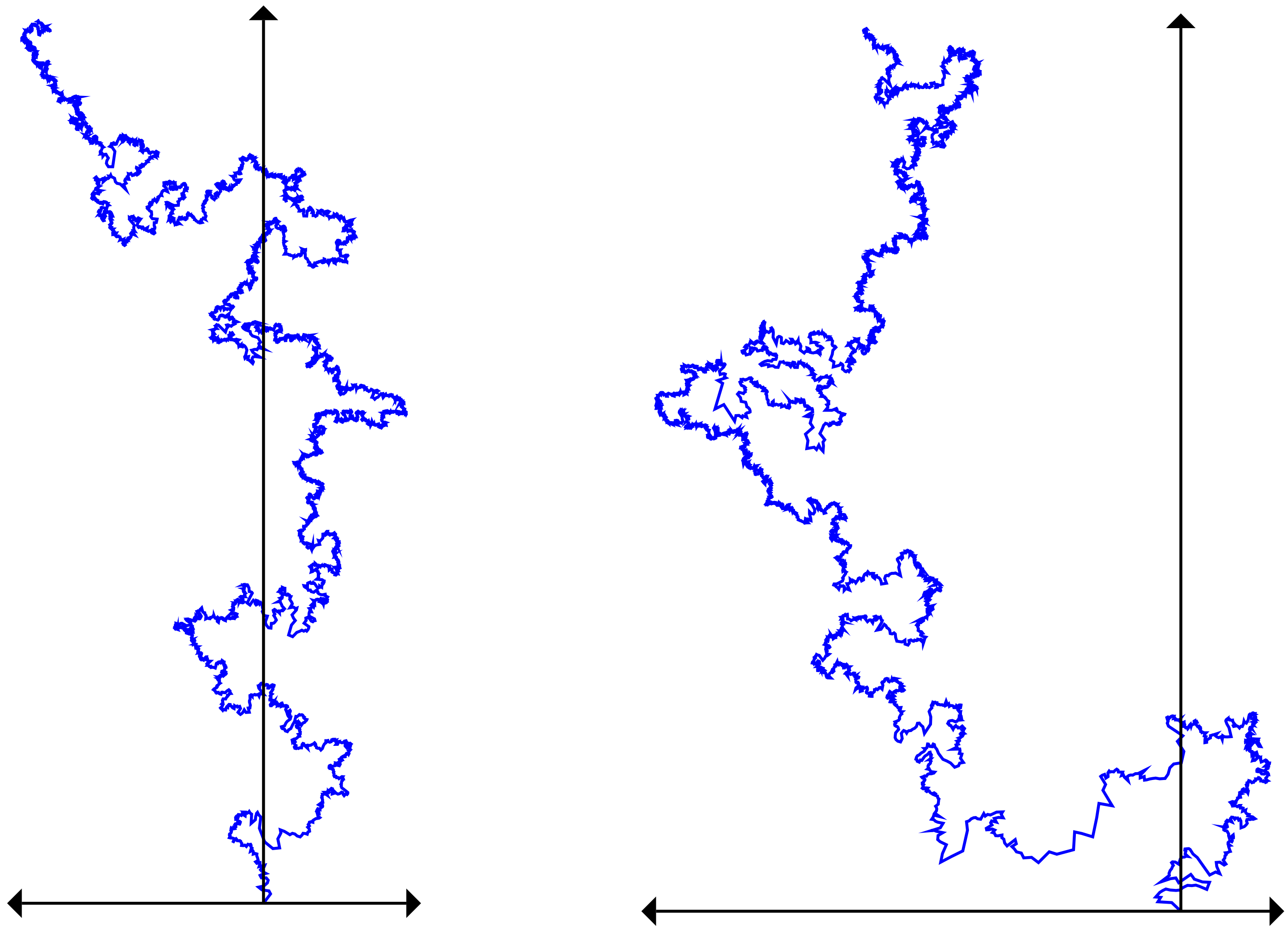
Review of SLE (cont)

The question is now whether there exists a curve associated with the maps g_t.

- If $0 < \kappa \le 4$, then there exists a random simple curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0,\infty) \subset \mathbb{H}$, i.e., the curve $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$ never re-visits $\mathbb{R}$. As well, the maps g_t obtained by solving ($*$) are conformal transformations of $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$. For this range of κ, our intuition matches the theory!
- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$. These curves have double points and they do hit $\mathbb{R}$, but they never cross themselves! As such, $\mathbb{H} \setminus \gamma(0,t]$ is not simply connected. However, $\mathbb{H} \setminus \gamma(0,t]$ does have a unique connected component containing ∞. This is $\mathbb{H}_t$ and the maps g_t are conformal transformations of $\mathbb{H}_t$ onto $\mathbb{H}$. We think of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ where K_t is the hull of $\gamma(0,t]$ visualized by taking $\gamma(0,t]$ and filling in the holes.
- For $\kappa \ge 8$, there exists a random curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ which is space-filling! Furthermore, it has double points, but does not cross itself!

As a result, we also refer to the curve γ as chordal SLE_κ. SLE paths are extremely rough: the Hausdorff dimension of a chordal SLE_κ path is $\min\{1 + \kappa/8, 2\}$.


$\kappa = 1$
$\kappa = 2$



$\kappa = \frac{8}{3}$ $\kappa = 3$

Review of SLE (cont)

Since there exists a curve γ associated with the maps g_t, it is possible to reparametrize it.

It can be shown that if U_t is a standard one-dimensional Brownian motion, then the solution to the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2/\kappa}{g_t(z) - U_t} = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

is chordal SLE_κ parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t/\kappa = at$.

Finally, chordal SLE as we have defined it can also be thought of as a measure on paths in the upper half plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

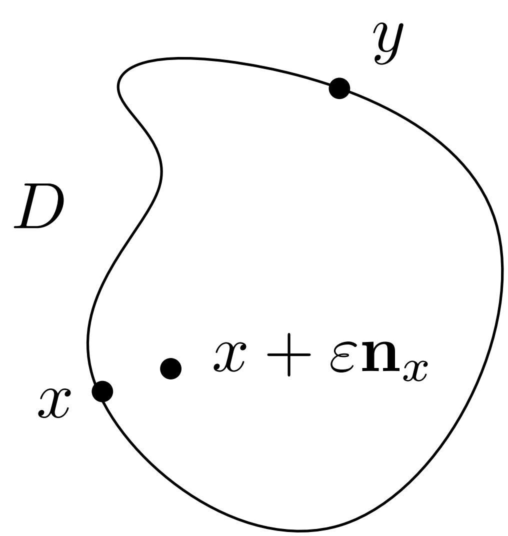
SLE is conformally invariant and so we can define chordal SLE_κ in any simply connected domain D connecting distinct boundary points z and w to be the image of chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation from $\mathbb{H}$ onto D sending $0 \mapsto z$ and $\infty \mapsto w$.

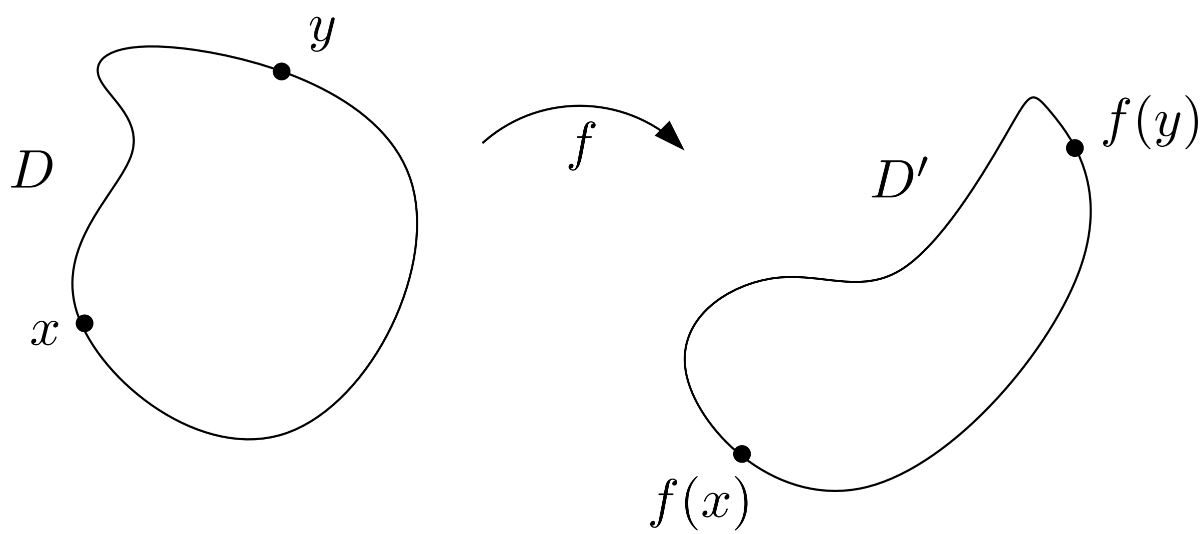
Excursion Poisson Kernel

Suppose that $D \subset \mathbb{C}$ is a simply connected Jordan domain and that ∂D is locally analytic at x and y. The excursion Poisson kernel is defined as

$$H_{\partial D}(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y)$$

where $H_D(z, y)$ for $z \in D$ is the usual Poisson kernel, and $\mathbf{n}_x$ is the unit normal at x pointing into D.





Proposition: If $f : D \to D'$ is a conformal transformation where $D' \subset \mathbb{C}$ is also a simply connected Jordan domain, and $\partial D'$ is locally analytic at $f(x)$, $f(y)$, then

$$H_{\partial D}(x, y) = |f'(x)||f'(y)|H_{\partial D'}(f(x), f(y)).$$

Example: Unit disk $\mathbb{D}$: $H_{\partial \mathbb{D}}(x, y) = \dfrac{1}{\pi\,|y - x|^2} = \dfrac{1}{2\pi(1 - \cos(\arg y - \arg x))}$.

Example: Upper half plane $\mathbb{H}$: $H_{\partial \mathbb{H}}(x, y) = \dfrac{1}{\pi\,(y - x)^2}$.

SLE as a finite measure on paths

SLE is conformally invariant and so we can define chordal SLE_κ in any simply connected domain D connecting distinct boundary points z and w to be the image of chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation from $\mathbb{H}$ onto D sending $0 \mapsto z$ and $\infty \mapsto w$.

Let $\mu_D^{\#}(z,w)$ denote the chordal SLE_κ probability measure on paths in D from z to w.

Define the finite measure

$$Q_D(z,w) = H(D;z,w)\mu_D^{\#}(z,w)$$

where $H(D;z,w)$ is defined for the upper half plane $\mathbb{H}$ by setting

$$H(\mathbb{H};0,\infty) = 1 \quad \text{and} \quad H(\mathbb{H};x,y) = \frac{1}{|y-x|^{2b}}$$

and for other simply connected domains D by conformal covariance

$$H(D;z,w) = |f'(z)|^b \, |f'(w)|^b H(D';f(z),f(w))$$

where $f: D \to D'$ is a conformal transformation (assuming appropriate smoothness) and $b > 0$ is a parameter.

SLE as a finite measure on paths (cont)

If we choose $b = \frac{6-\kappa}{2\kappa}$, then for $b \geq \frac{1}{4}$ (i.e., $0 < \kappa < 4$), the measure $Q_D(z,w)$ satisfies:

- **Conformal covariance.** If $f : D \to f(D)$ is a conformal transformation and $f(D)$ is analytic at $f(\mathbf{z})$, $f(\mathbf{w})$, then

$$f \circ Q_D(z,w) = |f'(z)|^b \, |f'(w)|^b \, Q_{f(D)}(f(z), f(w))$$

- **Boundary perturbation.** If $D \subset D'$ and ∂D, $\partial D'$ agree near z, w, then

$$Y_{D,D'}(z,w)(\gamma) = \frac{dQ_D(z,w)}{dQ_{D'}(z,w)}(\gamma) = 1\{\gamma \subset D\} e^{-c\Theta/2}$$

  where Θ is the measure of the set of Brownian loops in D' that intersect both γ and D, and $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$.

- In particular, if $f : D' \to f(D')$ is a conformal transformation, then

$$\frac{dQ_D(z,w)}{dQ_{D'}(z,w)} = \frac{dQ_{f(D)}(f(z), f(w))}{dQ_{f(D')}(f(z), f(w))}.$$

The Basic Setup for Multiple Paths

– $D \subset \mathbb{C}$ simply connected, ∂D Jordan

– $z_1, \ldots, z_n, w_n, \ldots, w_1$ distinct points ordered counterclockwise on ∂D

– write $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$

– fix a parameter $b > 0$ (boundary scaling exponent or boundary conformal weight)

Goal: To define a measure

$$Q_{D,b,n}(\mathbf{z}, \mathbf{w})$$

on mutually avoiding n-tuples $(\gamma^1, \ldots, \gamma^n)$ of simple paths in D, and satisfying certain properties:

(1) conformal covariance (2) boundary perturbation

(3) cascade relation (4) Markov property

Note that $\gamma^i : [0, t^i] \to \mathbb{C}$ with $\gamma^i(0) = z_i$, $\gamma^i(t^i) = w_i$, $\gamma(0, t^i) \subset D$.

Conformal Covariance

If D is analytic at $\mathbf{z}$, $\mathbf{w}$, then $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is a non-zero, finite measure supported on n-tuples $(\gamma^1,\ldots,\gamma^n)$ where γ^j is a simple curve in D connecting z_j and w_j and

$$\gamma^j \cap \gamma^k = \emptyset, \quad 1 \leq j < k \leq n.$$

Moreover, if $f : D \to f(D)$ is a conformal transformation and $f(D)$ is analytic at $f(\mathbf{z})$, $f(\mathbf{w})$, then

$$f \circ Q_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, Q_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})) \qquad (*)$$

where $f(\mathbf{z}) = (f(z_1),\ldots,f(z_n))$ and $f'(\mathbf{z}) = f'(z_1)\cdots f'(z_n)$.

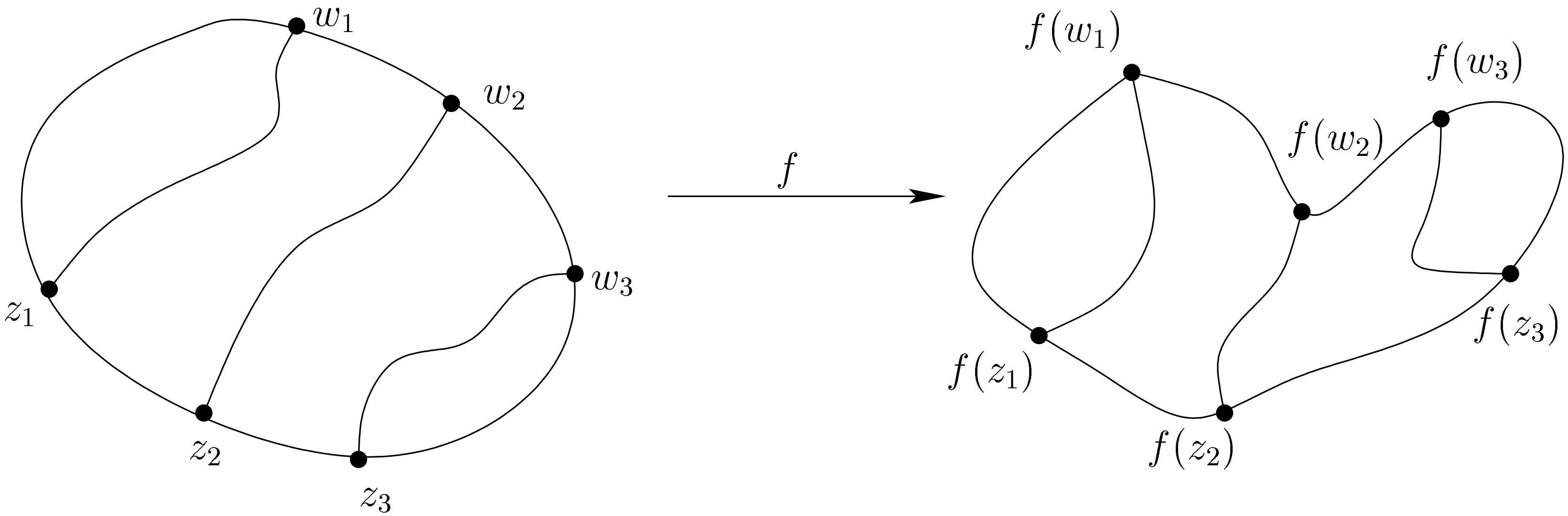


Figure 1: Conformal Covariance

Recall: $f \circ Q_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, Q_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w}))$ $(*)$

Write

$$Q_{D,b,n}(\mathbf{z},\mathbf{w}) = H_{D,b,n}(\mathbf{z},\mathbf{w}) \, \mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w}),$$

where $H_{D,b,n}(\mathbf{z},\mathbf{w}) = |Q_{D,b,n}(\mathbf{z},\mathbf{w})|$ and $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$ is a probability measure.

The conformal covariance condition $(*)$ then becomes the scaling rule for H,

$$H_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, H_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})),$$

and the conformal <u>invariance</u> rule for $\mu^{\#}$,

$$f \circ \mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w}) = \mu^{\#}_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})).$$

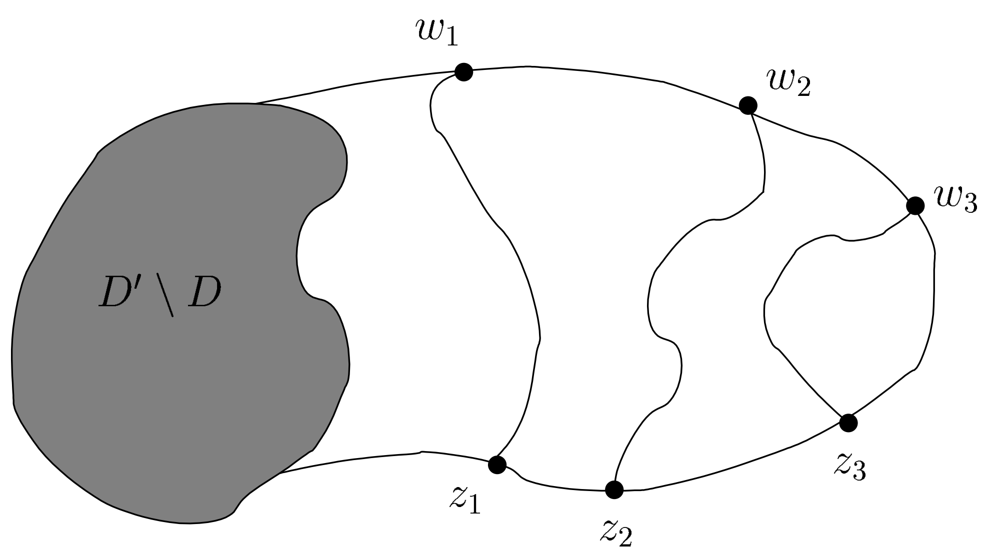
Since $\mu^{\#}$ is a conformal invariant, we can define $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$ even if the boundaries are not smooth at $\mathbf{z}$, $\mathbf{w}$.

Boundary Perturbation

Suppose $D \subset D'$ are Jordan domains and ∂D, $\partial D'$ agree and are analytic in neighbourhoods of $\mathbf{z}$, $\mathbf{w}$. Then $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is absolutely continuous with respect to $Q_{D',b,n}(\mathbf{z},\mathbf{w})$. Moreover, the Radon-Nikodym derivative

$$Y_{D,D',b,n}(\mathbf{z},\mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z},\mathbf{w})}{dQ_{D',b,n}(\mathbf{z},\mathbf{w})}$$

is a conformal invariant.



Recall: $D \subset D'$ and

$$Y_{D,D',b,n}(\mathbf{z},\mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z},\mathbf{w})}{dQ_{D',b,n}(\mathbf{z},\mathbf{w})}$$

Saying that $Y_{D,D',b,n}(\mathbf{z},\mathbf{w})$ is a conformal invariant means that if $f : D' \to f(D')$ is a conformal map that extends analytically in neighbourhoods of $\mathbf{z}$, $\mathbf{w}$, then

$$Y_{f(D),f(D'),b,n}(f(\mathbf{z}),f(\mathbf{w}))(f \circ \bar{\gamma}) = Y_{D,D',b,n}(\mathbf{z},\mathbf{w})(\bar{\gamma}), \tag{$*$}$$

where $\bar{\gamma} = (\gamma^1, \ldots, \gamma^n)$ and $f \circ \bar{\gamma} = (f \circ \gamma^1, \ldots, f \circ \gamma^n)$.

As with $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$, the last condition $(*)$ implies that $Y_{D,D',b,n}(\mathbf{z},\mathbf{w})$ is well-defined even if the boundaries are not smooth at $\mathbf{z},\mathbf{w}$.

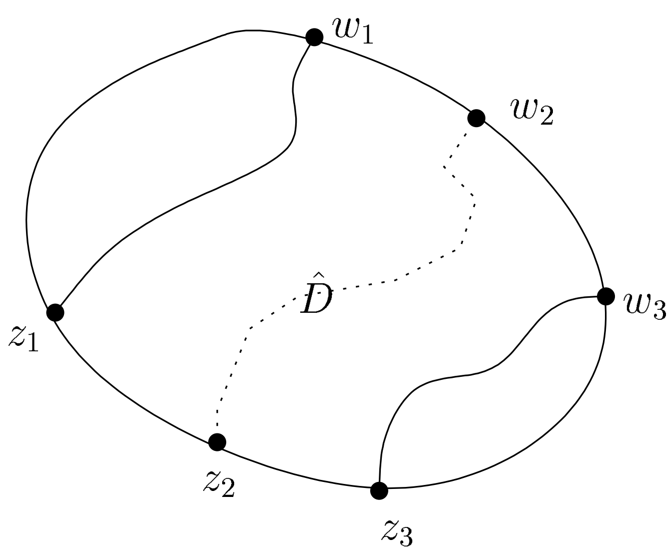
Cascade Relation

Let

$$\hat{\mathbf{z}} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n), \quad \hat{\mathbf{w}} = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n),$$

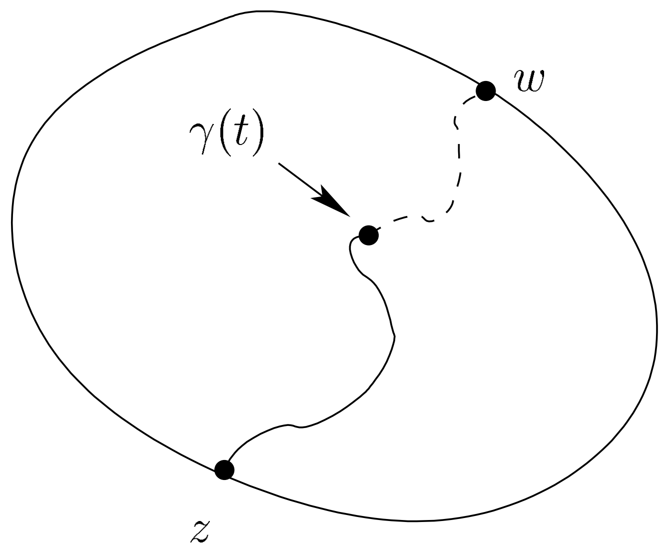
$$\hat{\boldsymbol{\gamma}} = (\gamma^1, \ldots, \gamma^{j-1}, \gamma^{j+1}, \ldots, \gamma^n).$$

The marginal distribution on $\hat{\boldsymbol{\gamma}}$ induced by $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ is absolutely continuous with respect to $Q_{D,b,n-1}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$ with Radon-Nikodym derivative $H_{\hat{D},b,1}(z_j, w_j)$. Here $\hat{D}$ is the subdomain of $D \setminus \hat{\boldsymbol{\gamma}}$ whose boundary includes z_j, w_j.



Markov Property

In the measure $\mu^{\#}_{D,b,1}(z,w)$, the conditional distribution on γ given an initial segment $\gamma[0,t]$ is $\mu^{\#}_{D\setminus\gamma[0,t],b,1}(\gamma(t),w)$.



Note: We have stated this condition in a way that does not use two dimensions and conformal invariance.

Existence of the Configurational Measure

Theorem (K-Lawler): For any $b \geq \frac{1}{4}$, there exists a family of measures $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves satisfying

- conformal covariance
- boundary perturbation
- cascade relation
- Markov property

Moreover, the simple curve γ^i is a chordal SLE$_\kappa$ from z_i to w_i in D where

$$\kappa = \frac{6}{2b+1}.$$

Note: $b \geq \frac{1}{4} \longleftrightarrow 0 < \kappa \leq 4$

The Partition Function for Two Paths

By conformal invariance, it suffices to work in $D = \mathbb{H}$.

If $0 < x_1 < \cdots < x_n < y_n < \cdots < y_1 < \infty$, let

$$H^*_{\mathbb{H},b,n}(\mathbf{x},\mathbf{y}) = \lim_{w\to\infty} w^{2b}\, H_{\mathbb{H},b,n+1}((0,\mathbf{x}),(w,\mathbf{y})).$$

Proposition: If $b \geq 1/4$ and $n+1 = 2$, then

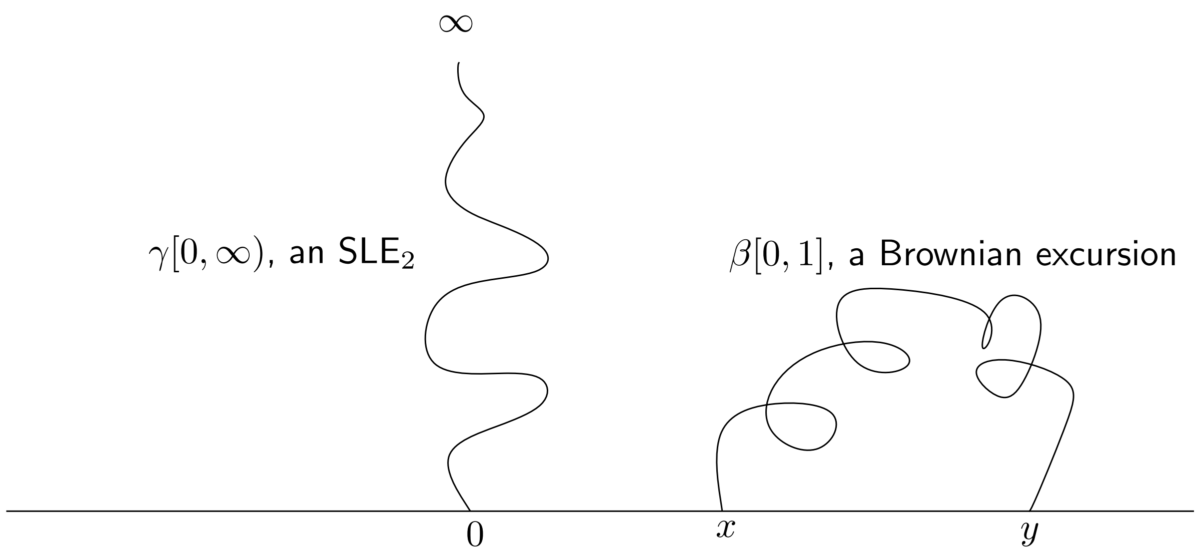
$$H^*_{\mathbb{H},b,1}(x,y) = (y-x)^{-2b}\, \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\, F(2a, 1-2a, 4a; x/y)$$

where F denotes the hypergeometric function and $a = \dfrac{2}{\kappa} = \dfrac{2b+1}{3}$.

Note: This result first appeared in J. Dubédat, and was derived non-rigorously by M. Bauer, D. Bernard, and K. Kytölä. Our configurational approach provides another rigorous derivation.

Fomin's identity: The Motivating Question

What is the probability that $\gamma[0,\infty)$, a chordal SLE_2 from 0 to ∞ in the upper half plane $\mathbb{H}$, and $\beta[0,t_\beta]$, a Brownian excursion from x to y in $\mathbb{H}$, do not intersect (with $0 < x < y < \infty$)?



Question: What is $\mathbf{P}\{\,\gamma[0,\infty)\cap\beta[0,t_\beta]=\emptyset\,\}$?

Motivation

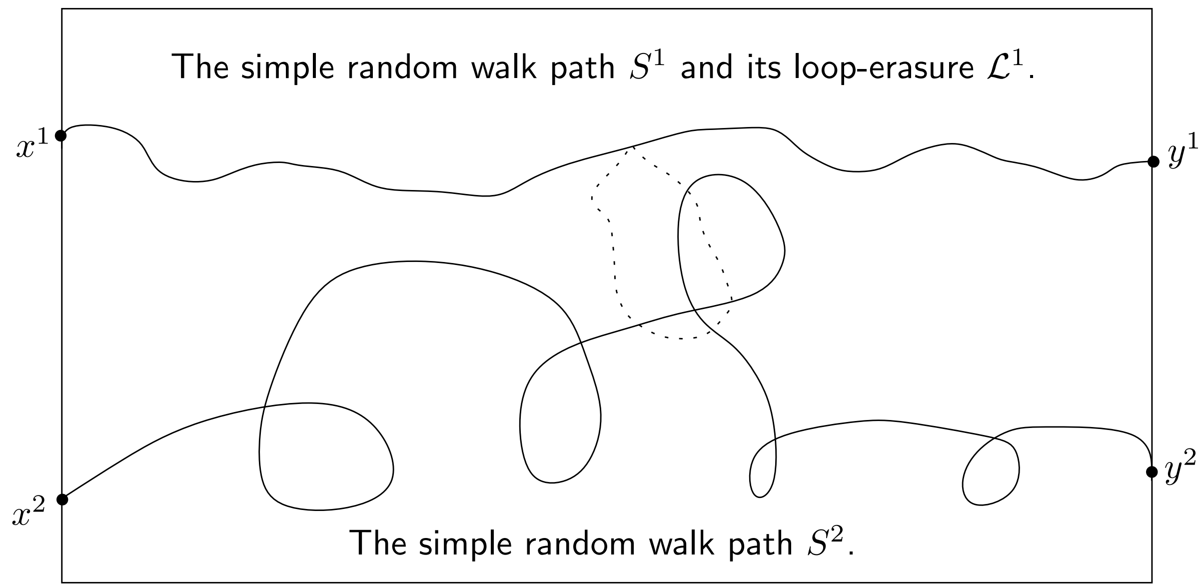
The motivation for asking this question is that the probability under consideration is the natural continuous analogue of the probability that arises in Fomin's identity. In fact, Fomin's original identity expressed the probability of a particular functional of loop-erased random walk in terms of the determinant of the hitting matrix for simple random walk, and in that work he conjectured that this identity holds for continuous processes:

> "... we do not need the notion of loop-erased Brownian motion. Instead, we discretize the model, compute the probability, and then pass to the limit."

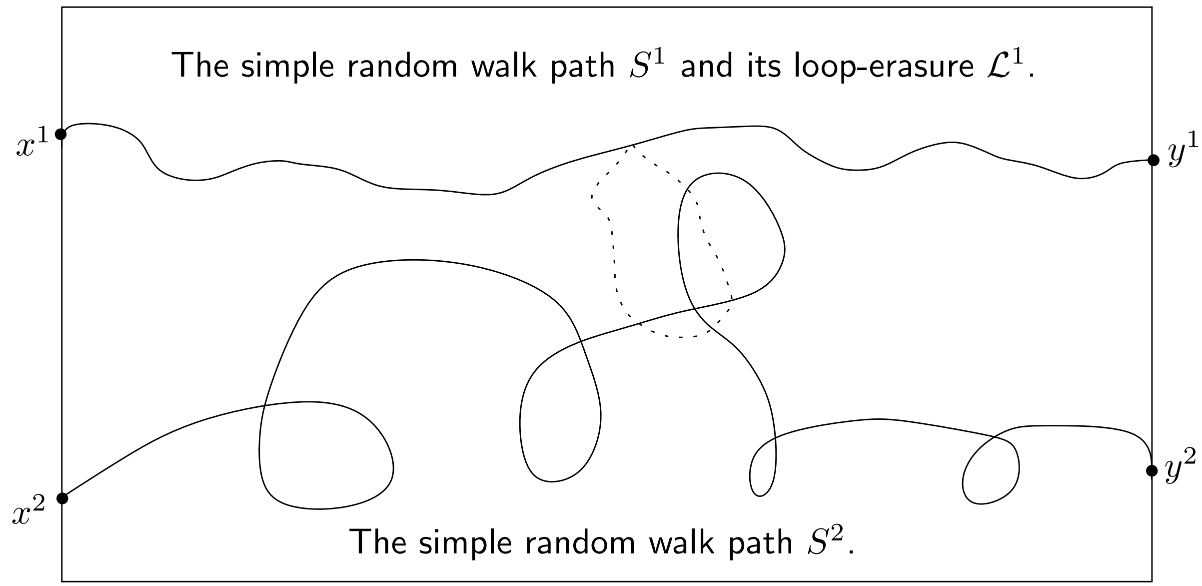
Fomin's identity for two paths

Fomin's original identity actually holds in much more generality and may be viewed as an extension of the Karlin-McGregor formula.

The version we state is for the special case of two simple random walk paths in a finite, simply connected subset $A \subset \mathbb{Z}^2$ connecting pairs of boundary points x^1, x^2, y^2, y^1 ordered counterclockwise around ∂A.



Note: In this example, $\mathcal{L}^1 \cap S^2 = \emptyset$ although $S^1 \cap \mathcal{L}^2 \neq \emptyset$.



Theorem (Fomin): If $\mathcal{L}^1$ is the path of a loop-erased random walk excursion from x^1 to y^1, and S^2 is the path of a simple random walk excursion from x^2 to y^2, then

$$\mathbf{P}\{\,\mathcal{L}^1 \cap S^2 = \emptyset\,\} = \frac{\det \mathbf{h}_{\partial A}(\mathbf{x},\mathbf{y})}{h_{\partial A}(x^1,y^1)\,h_{\partial A}(x^2,y^2)}$$

$$= \frac{h_{\partial A}(x^1,y^1)\,h_{\partial A}(x^2,y^2) - h_{\partial A}(x^1,y^2)\,h_{\partial A}(x^2,y^1)}{h_{\partial A}(x^1,y^1)\,h_{\partial A}(x^2,y^2)}$$

where $h_{\partial A}(x,y) := \mathbf{P}^x\{S_{\tau_A} = y,\ S_1 \in A\}$ is the discrete excursion Poisson kernel.

The non-intersection probability of SLE_2 *and Brownian motion*

Theorem (K-Lawler): If $0 < x < y < \infty$ are real numbers, $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE_2 from 0 to ∞ in $\mathbb{H}$, and $\beta : [0, t_\beta] \to \overline{\mathbb{H}}$ is a Brownian excursion from x to y in $\mathbb{H}$, then

$$\mathbf{P}\{\, \gamma[0,\infty) \cap \beta[0, t_\beta] = \emptyset \,\} = \frac{\det \mathbf{H}_{\partial\mathbb{D}}(f(\mathbf{x}), f(\mathbf{y}))}{H_{\partial\mathbb{D}}(f(0), f(\infty))\, H_{\partial\mathbb{D}}(f(x), f(y))} \qquad (*)$$

where $f : \mathbb{H} \to \mathbb{D}$ is a conformal transformation.

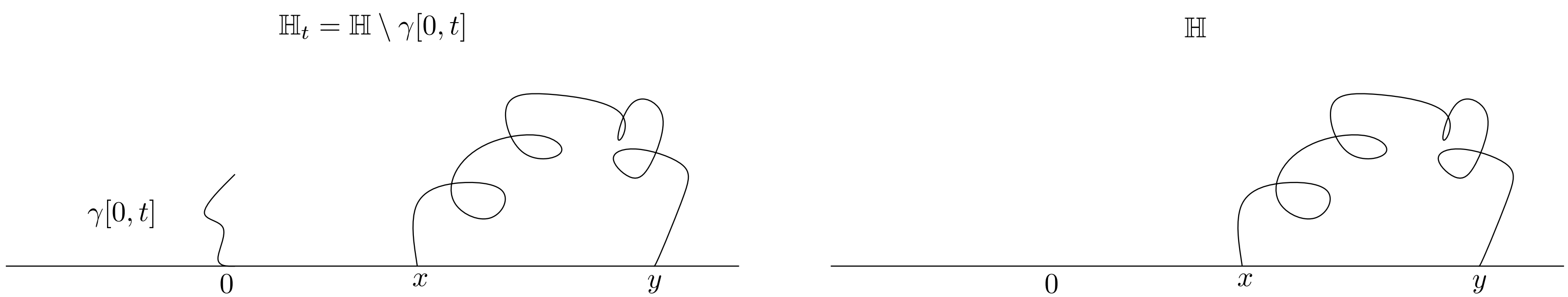
Strategy for the Proof

Our strategy for establishing this result will be as follows. We will first determine an explicit expression for $\mathbf{P}\{\, \gamma[0,\infty) \cap \beta[0, t_\beta] = \emptyset \,\}$, and we will then show that this explicit expression is the same as the right side of $(*)$.

Proof

For $0 < t < \infty$, let $\mathbb{H}_t$ denote the slit-plane $\mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t]$ so that

$$\mathbf{P}\{\gamma[0,t] \cap \beta[0,t_\beta] = \emptyset\} = \mathbb{E}^{x,y}\left[\frac{H_{\partial \mathbb{H}_t}(x,y)}{H_{\partial \mathbb{H}}(x,y)}\right].$$



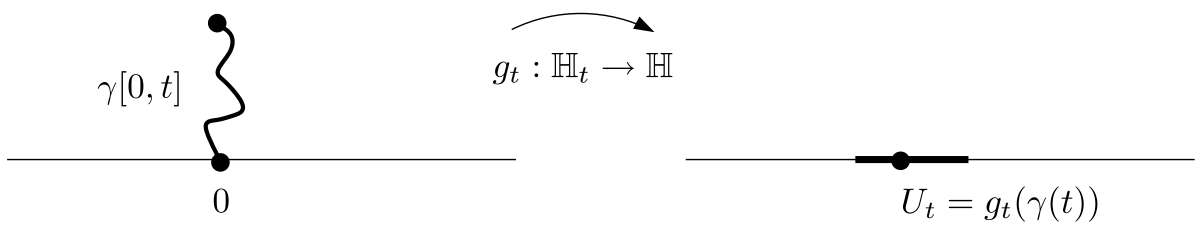
Letting $t \to \infty$ implies

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset\} = \mathbb{E}^{x,y}\left[\lim_{t\to\infty} \frac{H_{\partial \mathbb{H}_t}(x,y)}{H_{\partial \mathbb{H}}(x,y)}\right].$$

Let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the unique conformal transformation satisfying the hydrodynamic normalization $g_t(z) - z = o(1)$ as $z \to \infty$ so that g_t satisfies the chordal Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{1}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = -B_t$ is a standard Brownian motion.



We now map $\mathbb{H}_t$ to $\mathbb{H}$ by g_t and use conformal covariance to conclude that

$$H_{\partial \mathbb{H}_t}(x, y) = g_t'(x) g_t'(y) H_{\partial \mathbb{H}}(g_t(x), g_t(y))$$

and so

$$\frac{H_{\partial \mathbb{H}_t}(x, y)}{H_{\partial \mathbb{H}}(x, y)} = \frac{g_t'(x) g_t'(y) H_{\partial \mathbb{H}}(g_t(x), g_t(y))}{H_{\partial \mathbb{H}}(x, y)} = (y - x)^2 \cdot \frac{g_t'(x) g_t'(y)}{(g_t(y) - g_t(x))^2}$$

where the last equality follows from the explicit form of $H_{\partial \mathbb{H}}$.

Recall: $\dfrac{H_{\partial \mathbb{H}_t}(x,y)}{H_{\partial \mathbb{H}}(x,y)} = (y-x)^2 \cdot \dfrac{g_t'(x) g_t'(y)}{(g_t(y)-g_t(x))^2}$

Let

$$J_t := \frac{g_t'(x) g_t'(y)}{(g_t(y)-g_t(x))^2} \quad \text{and set} \quad J_\infty := \lim_{t\to\infty} J_t.$$

Let $P(x,y) := \mathbf{P}\{\gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset\}$ so that

$$P(x,y) = (y-x)^2 \mathbb{E}^{x,y}\left[\lim_{t\to\infty} \frac{g_t'(x) g_t'(y)}{(g_t(y)-g_t(x))^2}\right] = (y-x)^2 \mathbb{E}^{x,y}[J_\infty].$$

In order to determine $P(x,y)$, we will derive and solve an ODE for it.

Let $X_t := g_t(x) + B_t$ and $Y_t := g_t(y) + B_t$ so that

$$dX_t = \frac{1}{X_t}dt + dB_t \quad \text{and} \quad dY_t = \frac{1}{Y_t}dt + dB_t.$$

Some routine calculations give

$$\frac{\partial}{\partial t} \log g_t'(x) = -\frac{1}{X_t^2}, \; \frac{\partial}{\partial t} \log g_t'(y) = -\frac{1}{Y_t^2}, \text{ and } \frac{\partial}{\partial t} \log(g_t(y) - g_t(x)) = -\frac{1}{X_t\, Y_t},$$

and so we see that

$$J_t = J_0 \, \exp\left\{\int_0^t \partial_s[\log J_s] \, ds\right\} = \frac{1}{(y-x)^2} \, \exp\left\{-\int_0^t \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2 \, ds\right\}.$$

Hence, putting things together we find

$$P(x, y) = \mathbb{E}^{x,y}\left[\exp\left\{-\int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2 \, ds\right\}\right].$$

It now follows from the usual Markov property that $J_t\, P(X_t, Y_t)$ is a martingale. That is, if $M_t := \mathbb{E}^{x,y}[J_\infty | \mathcal{F}_t]$ so that M_t is a martingale, then

$$\begin{aligned}
M_t &= \mathbb{E}^{x,y}\left[\frac{1}{(y-x)^2}\ \exp\left\{-\int_0^\infty \left(\frac{1}{X_s}-\frac{1}{Y_s}\right)^2\, ds\right\}\ \Big|\ \mathcal{F}_t\right] \\
&= \frac{1}{(y-x)^2}\exp\left\{-\int_0^t \left(\frac{1}{X_s}-\frac{1}{Y_s}\right)^2\, ds\right\}\cdot \\
&\qquad\qquad \cdot\, \mathbb{E}^{x,y}\left[\exp\left\{-\int_t^\infty \left(\frac{1}{X_s}-\frac{1}{Y_s}\right)^2\, ds\right\}\ \Big|\ \mathcal{F}_t\right] \\
&= J_t\, P(X_t, Y_t).
\end{aligned}$$

Itô's formula now implies that

$$-\left(\frac{1}{x}-\frac{1}{y}\right)^2\, P + \frac{1}{x}\,\frac{\partial P}{\partial x} + \frac{1}{y}\,\frac{\partial P}{\partial y} + \frac{1}{2}\,\frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\,\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial x \partial y} = 0.$$

Since the probability in question only depends on the ratio x/y, we see that $P(x,y) = \phi(x/y)$ for some function ϕ. Thus, letting $u = x/y$ and noting that $0 < u < 1$, we find

$$u^2\,(1-u)\,\phi''(u) + 2\,u\,\phi'(u) - 2(1-u)\,\phi(u) = 0.$$

The constraint $0 < u < 1$ allows us to consider $\psi(u) := u^{-1}(1-u)^{-3}\phi(u)$ which satisfies the ODE

$$u\,(1-u)\,\psi''(u) + (4-8u)\,\psi'(u) - 10\,\psi(u) = 0.$$

This is the well-known hypergeometric differential equation, and so

$$\psi(u) = C_1\,\frac{2-u}{(1-u)^3} + C_2\,\frac{1-2u}{u^3(1-u)^3}$$

which implies that

$$\phi(u) = C_1 u(2-u) + C_2 u^{-2}(1-2u).$$

However, physical considerations dictate that $\phi(u) \to 0$ as $u \to 0+$ and $\phi(u) \to 1$ as $u \to 1-$, and so $C_2 = 0$ and $C_1 = 1$.

Thus, $\phi(u) = u(2-u)$ and so we find

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset\} = P(x,y) = \phi(x/y) = \frac{x}{y}\left(2 - \frac{x}{y}\right).$$

As already noted, the probability in question only depends on the ratio x/y, and so it suffices without loss of generality to assume that $0 < x < 1$ and $y = 1$.

Furthermore, we may assume that the conformal transformation $f : \mathbb{H} \to \mathbb{D}$ is

$$f(z) = \frac{iz + 1}{z + i},$$

so that $f(0) = -i$, $f(y) = f(1) = 1$, $f(\infty) = i$, and

$$f(x) = \left(\frac{2x}{x^2 + 1}\right) + i\left(\frac{x^2 - 1}{x^2 + 1}\right) = \exp\left\{-i \arctan\left(\frac{1 - x^2}{2x}\right)\right\}.$$

Writing $f(x) = e^{i\theta}$, we find that

$$\begin{aligned}
&\frac{\det \mathbf{H}_{\partial\mathbb{D}}(f(\mathbf{x}), f(\mathbf{y}))}{H_{\partial\mathbb{D}}(f(0), f(\infty))\, H_{\partial\mathbb{D}}(f(x), f(y))} \\
&\qquad = \frac{H_{\partial\mathbb{D}}(-i, i)\, H_{\partial\mathbb{D}}(e^{i\theta}, 1) - H_{\partial\mathbb{D}}(-i, 1)\, H_{\partial\mathbb{D}}(e^{i\theta}, i)}{H_{\partial\mathbb{D}}(-i, i)\, H_{\partial\mathbb{D}}(e^{i\theta}, 1)} \\
&\qquad = \frac{2\cos\theta + \sin\theta - 1}{1 + \sin\theta}.
\end{aligned}$$

Since $\theta = -\arctan\left(\frac{1-x^2}{2x}\right)$ we see that $\cos\theta = \frac{2x}{x^2+1}$ and $\sin\theta = \frac{1-x^2}{x^2+1}$ which upon substitution gives

$$\frac{2\cos\theta + \sin\theta - 1}{1 + \sin\theta} = \frac{\frac{4x}{x^2+1} + \frac{1-x^2}{x^2+1} - 1}{1 + \frac{1-x^2}{x^2+1}} = \frac{4x - 2x^2}{2} = x(2-x).$$

Comparing this with our earlier result proves the theorem. $\square$

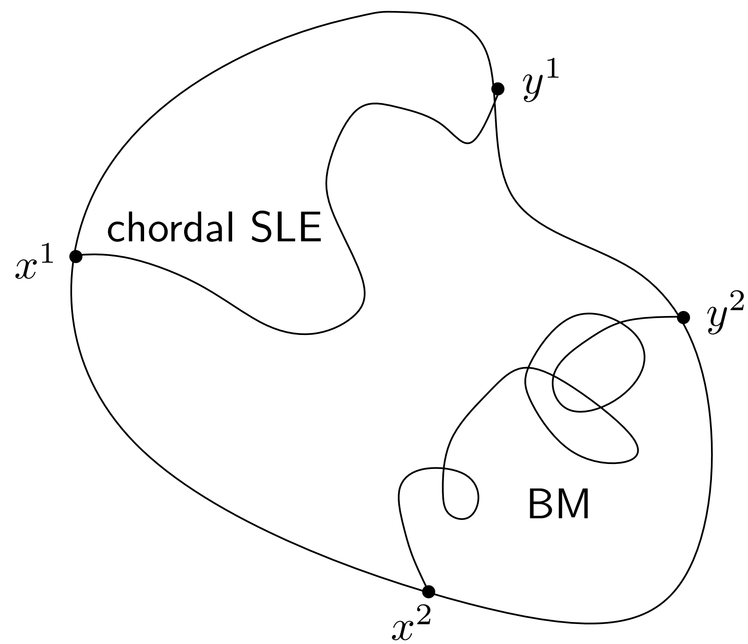
An Example

Example: Let $x = 1/2$ and $y = 1$. Then $f : \mathbb{H} \to \mathbb{D}$ has $f(0) = i$, $f(1) = 1$, $f(\infty) = -i$, and $f(1/2) = \exp\{-i \arctan(3/4)\}$. A simple calculation gives

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0, t_\beta] = \emptyset\} = \frac{2 \cdot \frac{4}{5} + \frac{3}{5} - 1}{1 + \frac{3}{5}} = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}.$$

Corollary

Suppose that $D \subset \mathbb{C}$ is a bounded, simply connected planar domain, and that x^1, x^2, y^2, y^1 are four points ordered counterclockwise around ∂D. The probability a chordal SLE_2 from x^1 to y^1 in D does not intersect a Brownian excursion from x^2 to y^2 in D is $\Phi(x^2)\left(2 - \Phi(x^2)\right)$ where $\Phi : D \to \mathbb{H}$ is the conformal transformation with $\Phi(x^1) = 0$, $\Phi(y^1) = \infty$, $\Phi(y^2) = 1$.



This statement can be easily modified to cover the case when D is unbounded and/or the case when ∞ is one of the boundary points.