Intersection probabilities for a chordal SLE path and a semicircle

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This talk is based on joint work with **Tom Alberts** of the Courant Institute.

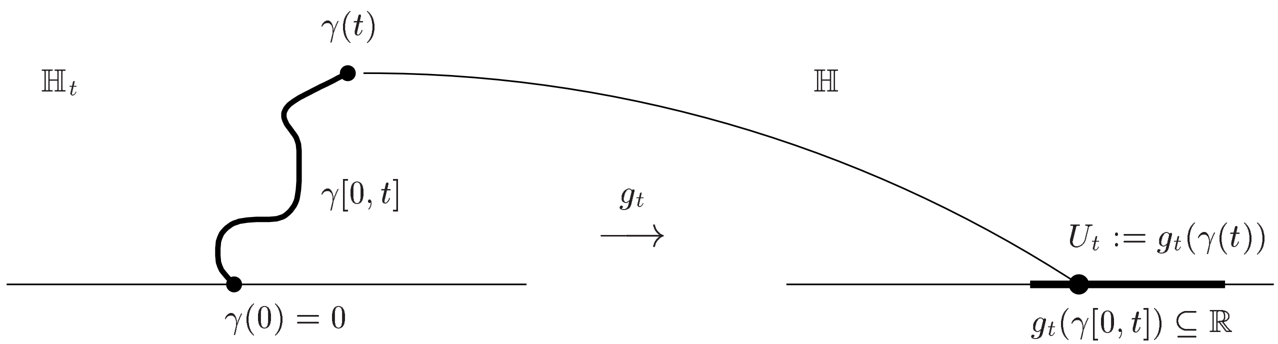
Review of SLE

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane, and consider a simple (non-self-intersecting) curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$.

For every fixed $t \geq 0$, the slit plane $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0, t]$ is simply connected and so by the Riemann mapping theorem, there exists a unique conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$ which can be expanded as

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty,$$

where $b(t) = \mathrm{hcap}(\gamma(0, t])$ is the **half-plane capacity** of γ up to time t.



It can be shown that there is a unique point $U_t \in \mathbb{R}$ for all $t \geq 0$ with $U_t := g_t(\gamma(t))$ and that the function $t \mapsto U_t$ is continuous.

Review of SLE (cont)

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty, \quad \mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t]$$

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by a PDE involving U_t.

This is due to C. Loewner (1923) who showed that if γ is a curve as above such that its half-plane capacity $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$, then for $z \in \mathbb{H}$ with $z \notin \gamma[0,\infty)$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the PDE

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Note that if $b(t) \in C^1$ is an increasing function, then we can reparametrize the curve γ so that $\mathrm{hcap}(\gamma(0,t]) = b(t)$. This is the so-called **parametrization by capacity**.

Review of SLE (cont)

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \qquad (*)$$

The obvious thing to do now is to start with a continuous function $t \mapsto U_t$ from $[0, \infty)$ to $\mathbb{R}$ and solve the Loewner equation for g_t.

Ideally, we would like to solve $(*)$ for g_t, define simple curves $\gamma(t)$, $t \geq 0$, by setting $\gamma(t) = g_t^{-1}(U_t)$, and have g_t map $\mathbb{H} \setminus \gamma(0, t]$ conformally onto $\mathbb{H}$.

Although this is the intuition, it is not quite precise because we see from the denominator on the right-side of $(*)$ that problems can occur if $g_t(z) - U_t = 0$.

Formally, if we let T_z be the supremum of all t such that the solution to $(*)$ is well-defined up to time t with $g_t(z) \in \mathbb{H}$, and we define $\mathbb{H}_t = \{z : T_z > t\}$, then g_t is the unique conformal transformation of $\mathbb{H}_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $t \to \infty$.

Review of SLE (cont)

The novel idea of Schramm was to take the continuous function U_t to be a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

The **chordal Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization** (or simply SLE_κ) is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t}\, g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}\, W_t}, \quad g_0(z) = z, \tag{LE}$$

where W_t is a standard one-dimensional Brownian motion.

Review of SLE (cont)

The question is now whether there exists a curve associated with the maps g_t.

- If $0 < \kappa \le 4$, then there exists a random simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$, i.e., the curve $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$ never re-visits $\mathbb{R}$. As well, the maps g_t obtained by solving $(*)$ are conformal transformations of $\mathbb{H} \setminus \gamma(0, t]$ onto $\mathbb{H}$. For this range of κ, our intuition matches the theory!
- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$. These curves have double points and they do hit $\mathbb{R}$, but they never cross themselves! As such, $\mathbb{H} \setminus \gamma(0, t]$ is not simply connected. However, $\mathbb{H} \setminus \gamma(0, t]$ does have a unique connected component containing ∞. This is $\mathbb{H}_t$ and the maps g_t are conformal transformations of $\mathbb{H}_t$ onto $\mathbb{H}$. We think of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ where K_t is the hull of $\gamma(0, t]$ visualized by taking $\gamma(0, t]$ and filling in the holes.
- For $\kappa \ge 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ which is space-filling! Furthermore, it has double points, but does not cross itself! As in the case $4 < \kappa < 8$, the maps g_t are conformal transformations of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ where K_t is the hull of $\gamma(0, t]$.

As a result, we also refer to the curve γ as chordal SLE_κ. SLE paths are extremely rough: the Hausdorff dimension of a chordal SLE_κ path is $\min\{1 + \kappa/8, 2\}$.

Review of SLE (cont)

Since there exists a curve γ associated with the maps g_t, it is possible to reparametrize it.

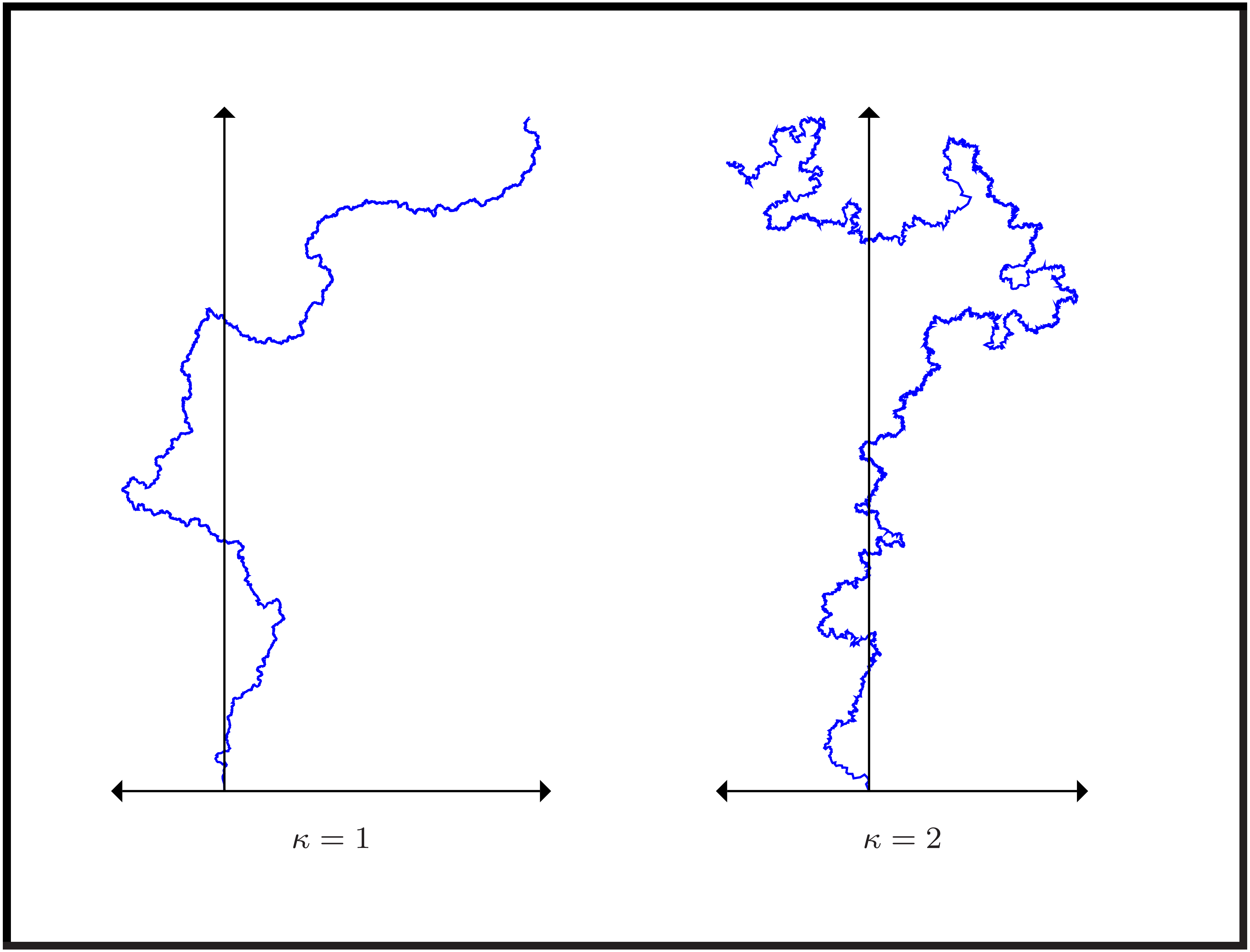
It can be shown that if U_t is a standard one-dimensional Brownian motion, then the solution to the initial value problem

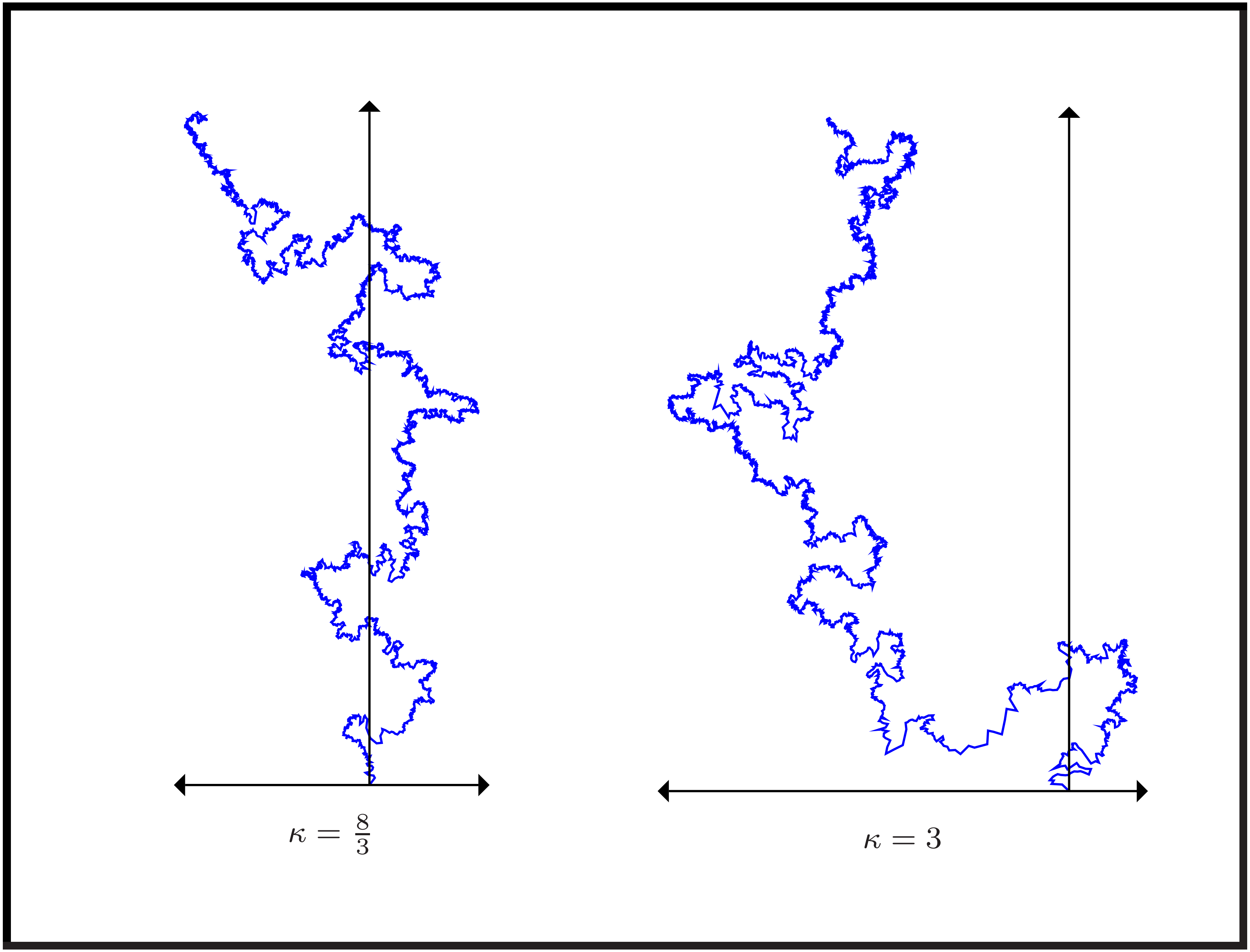
$$\frac{\partial}{\partial t} g_t(z) = \frac{2/\kappa}{g_t(z) - U_t} = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

is chordal SLE_κ parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t/\kappa = at$.

Finally, chordal SLE as we have defined it can also be thought of as a measure on paths in the upper half plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

SLE is conformally invariant and so we can define chordal SLE_κ in any simply connected domain D connecting distinct boundary points z and w to be the image of chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation from $\mathbb{H}$ onto D sending $0 \mapsto z$ and $\infty \mapsto w$.


$\kappa = 1$
$\kappa = 2$



$\kappa = \frac{8}{3}$
$\kappa = 3$

Our motivation

MK was interested in multiple SLEs and wanted to estimate the diameter of a chordal SLE path in $\mathbb{H}$ connecting the boundary points 0 and $x > 0$.

TA was interested in Hausdorff dimension and wanted to estimate the probability that a chordal SLE path in $\mathbb{H}$ connecting 0 and ∞ intersected a semicircle centred on the real line.

The two problems are the same.

Ideally, we hoped to determine these results asymptotically ($\sim$), but could only get them up to constants ($\asymp$).

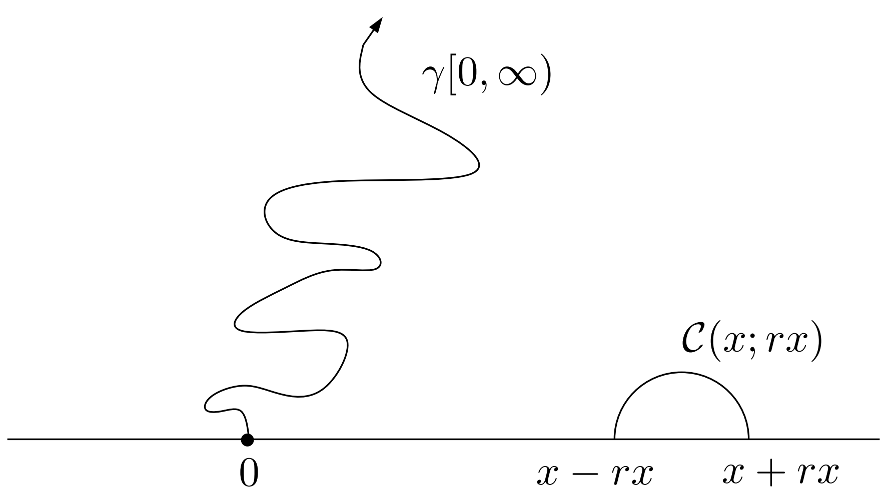
Note. $\sim$ implies $\asymp$ implies $\approx$.

The main estimate

Theorem. Let $x > 0$ be real, $0 < r \le 1/3$, and $\mathcal{C}(x; rx) = \{x + rxe^{i\theta} : 0 < \theta < \pi\}$ denote the semicircle of radius rx centred at x in the upper half plane, and suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞.

(a) If $0 < \kappa < 8$, then $P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \asymp r^{\frac{8-\kappa}{\kappa}}$.

(b) If $\kappa = 8/3$, then $P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} = 1 - (1 - r^2)^{5/8} \sim \frac{5}{8} r^2$.

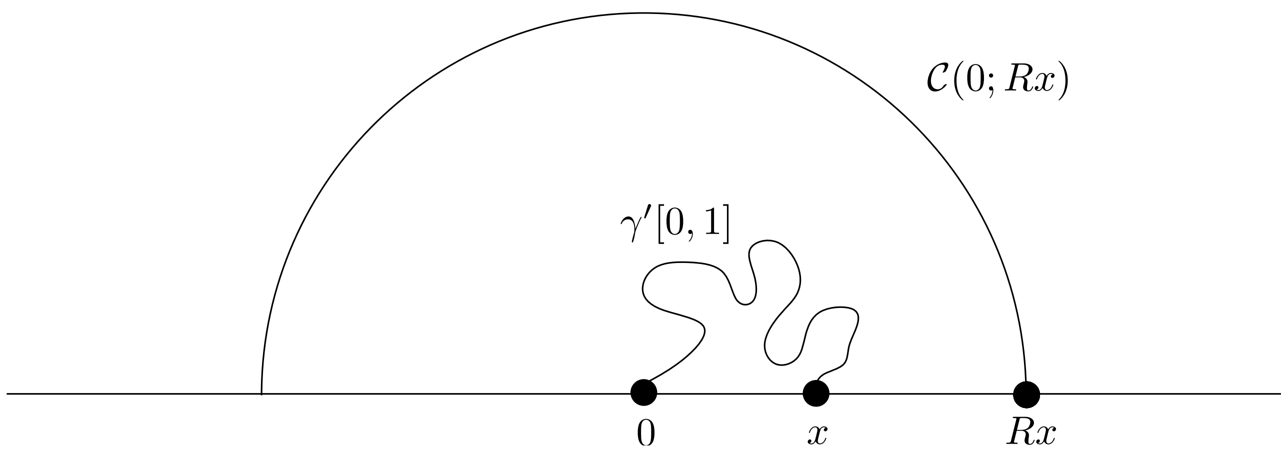


An equivalent formulation

Corollary. Let $x > 0$ be real, $R \geq 3$, and $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$ denote the circle of radius Rx centred at 0 in the upper half plane, and suppose that $\gamma' : [0,1] \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to x.

(a) If $0 < \kappa < 8$, then $P\{\gamma'[0,1] \cap \mathcal{C}(0; Rx) \neq \emptyset\} \asymp R^{\frac{\kappa-8}{\kappa}}$.

(b) If $\kappa = 8/3$, then $P\{\gamma'[0,1] \cap \mathcal{C}(0; Rx) \neq \emptyset\} = 1 - (1 - R^{-2})^{5/8} \sim \frac{5}{8} R^{-2}$.



Derivation of the corollary

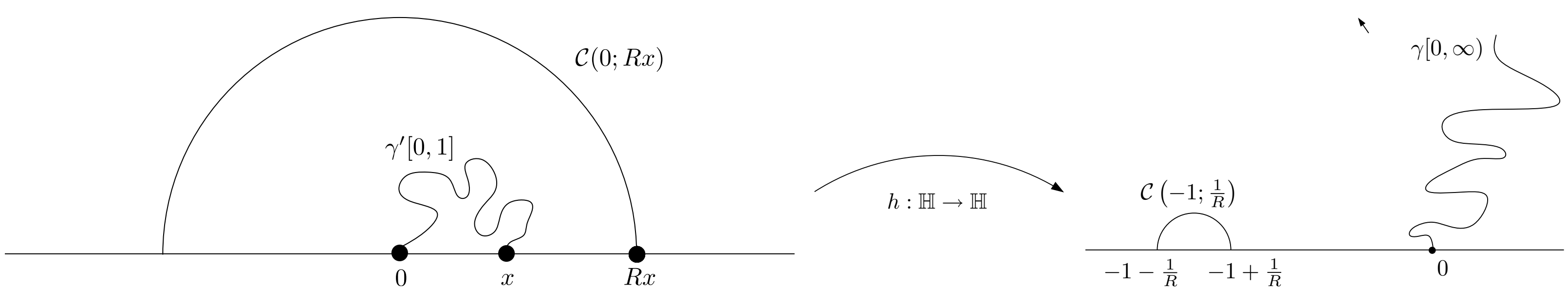
The idea is to determine the appropriate sequence of conformal transformations and use the conformal invariance of chordal SLE.

Suppose that $\gamma' : [0,1] \to \overline{\mathbb{H}}$ is an SLE$_\kappa$ in $\mathbb{H}$ from 0 to $x > 0$. Note that we are not interested in the parametrization of the SLE path, but only in the points visited by its trace. Suppose that $R \geq 3$, and consider $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$. For $z \in \mathbb{H}$, let

$$h(z) = \frac{R^2}{R^2 - 1} \frac{z}{x - z}$$

so that $h : \mathbb{H} \to \mathbb{H}$ is a conformal (Möbius) transformation with $h(0) = 0$ and $h(x) = \infty$. It is straightforward (though tedious) to verify that

$$h\left(\mathcal{C}(0; Rx)\right) = \mathcal{C}\left(-1; \frac{1}{R}\right).$$



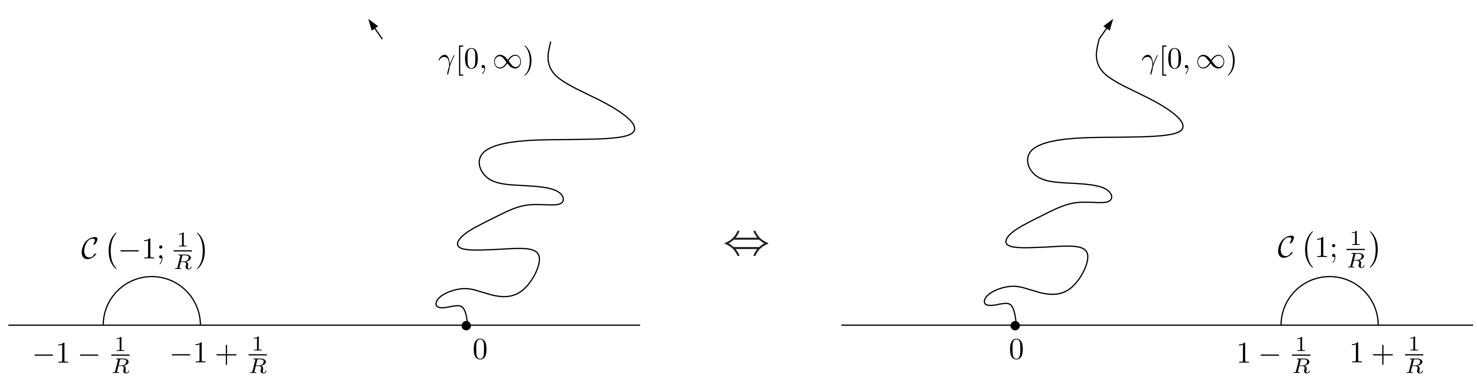
Derivation of the corollary (cont.)

If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then the conformal invariance of SLE implies that

$$P\{\gamma'[0,1] \cap \mathcal{C}(0;Rx) \neq \emptyset\} = P\{h(\gamma'[0,1]) \cap h(\mathcal{C}(0;Rx)) \neq \emptyset\}$$
$$= P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(-1, \frac{1}{R}\right) \neq \emptyset\right\}.$$

By the symmetry of SLE about the imaginary axis,

$$P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(-1, \frac{1}{R}\right) \neq \emptyset\right\} = P\left\{\gamma[0,\infty) \cap \mathcal{C}\left(1, \frac{1}{R}\right) \neq \emptyset\right\} \asymp R^{1-4a}.$$



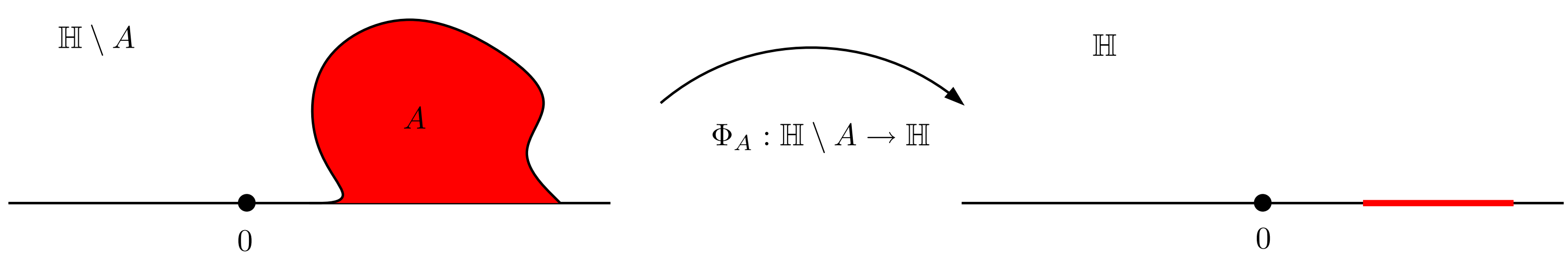
The $\kappa = 8/3$ case

The key fact that is needed is the restriction property of chordal $\text{SLE}_{8/3}$.

Fact. [Lawler-Schramm-Werner] If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal $\text{SLE}_{8/3}$ in $\mathbb{H}$ from 0 to ∞, and A is a bounded subset of $\mathbb{H}$ such that $\mathbb{H} \setminus A$ is simply connected, $A = \mathbb{H} \cap \overline{A}$, and $0 \notin \overline{A}$, then

$$P\{\gamma[0,\infty) \cap A = \emptyset\} = \left[\Phi'_A(0)\right]^{5/8}$$

where $\Phi_A : \mathbb{H} \setminus A \to \mathbb{H}$ is the unique conformal transformation of $\mathbb{H} \setminus A$ to $\mathbb{H}$ with $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$.

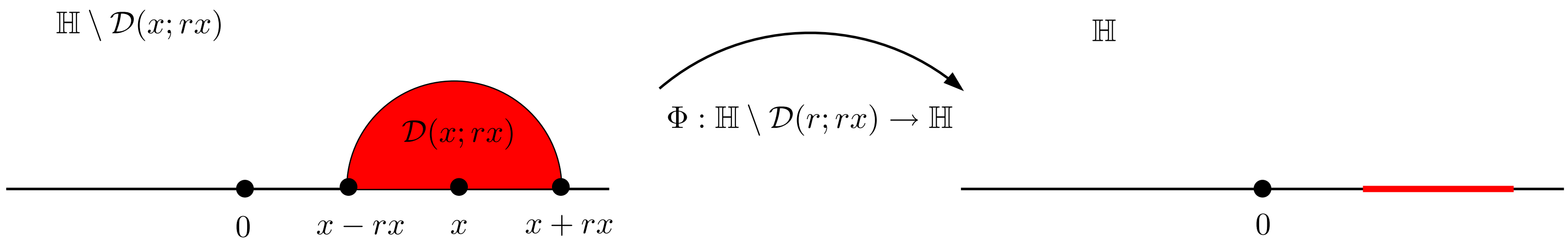


The $\kappa = 8/3$ case (cont.)

This implies that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;rx) = \emptyset\} = \left[\Phi'(0)\right]^{5/8}$$

where $\Phi = \Phi_{\mathcal{D}(x;rx)}(z)$ is the conformal transformation from $\mathbb{H} \setminus \mathcal{D}(x;rx)$ onto $\mathbb{H}$ with $\Phi(0) = 0$ and $\Phi(z) \sim z$ as $z \to \infty$.



In fact, the exact form of $\Phi(z)$ is given by

$$\Phi(z) = z + \frac{r^2x^2}{z - x} + r^2x.$$

Note that $\Phi(0) = 0$, $\Phi(\infty) = \infty$, and $\Phi'(\infty) = 1$. We calculate $\Phi'(0) = 1 - r^2$ and therefore conclude that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;rx) = \emptyset\} = (1 - r^2)^{5/8}.$$

Rephrasing the main estimate

Theorem. Let $x > 0$ be a fixed real number, and suppose $0 < \epsilon \le x/3$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$, then

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;\epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}$$

where $\mathcal{C}(x;\epsilon)$ is the semicircle of radius ϵ centred at x in the upper half plane.

Written in this form, it is seen to generalize the result of Rohde and Schramm who prove that for $4 < \kappa < 8$,

$$P\{\gamma[0,\infty) \cap [x-\epsilon, x+\epsilon] \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}.$$

An application

Let $0 < r \leq 1/3$, and suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$.

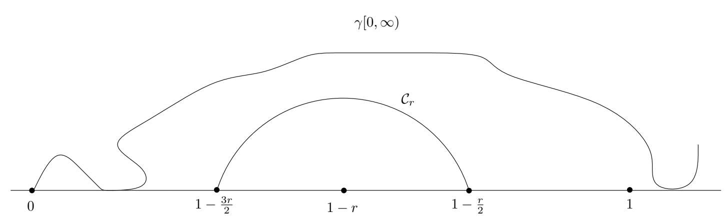
Theorem. There exist constants c'_a and c''_a such that

$$1 - c'_a r^{4a-1} \leq \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\} \leq \sup_{z \in \mathcal{C}_r} P\{T_z = T_1\} \leq 1 - c''_a r^{4a-1}$$

where

$$\mathcal{C}_r = \mathcal{C}\left(1 - r; \frac{r}{2}\right)$$

denotes the circle of radius $r/2$ centred at $1 - r$ in the upper half plane.



Corollary. There exist constants c'_a and c''_a such that

$$1 - c'_a r^{4a-1} \leq P\{T_z = T_1 \text{ for all } z \in \mathcal{C}_r\} \leq 1 - c''_a r^{4a-1}.$$

Proof of the application

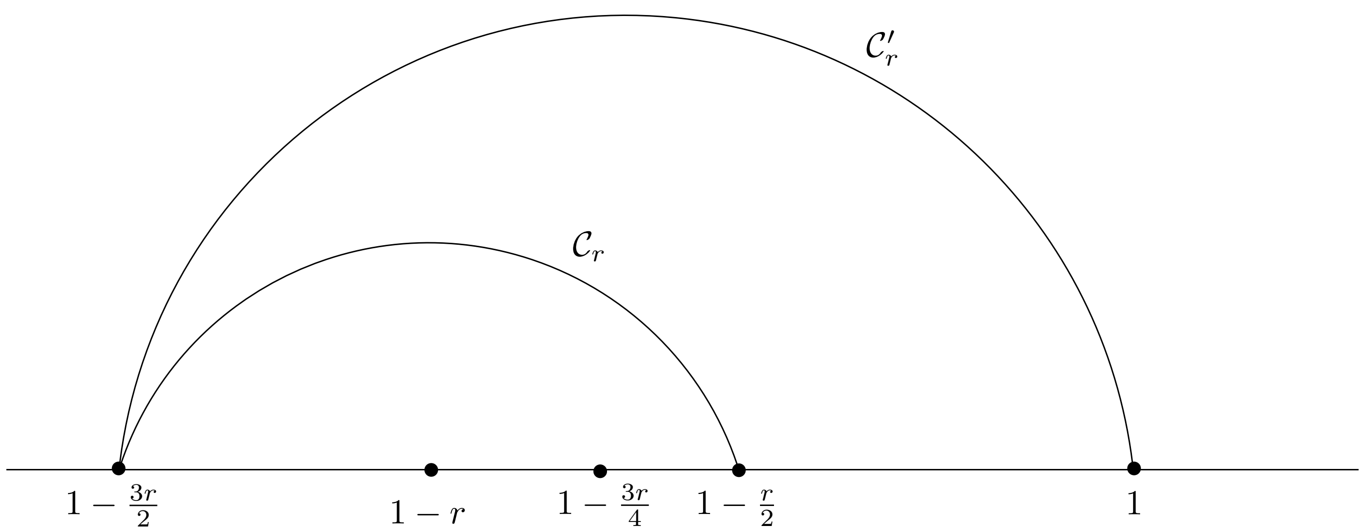
The proof follows by combining the main result with a method due to Dubédat.

Suppose that $0 < r \leq 1/3$ and consider the two semicircles

$$\mathcal{C}_r = \mathcal{C}\left(1 - r; \frac{r}{2}\right)$$

and

$$\mathcal{C}'_r = \mathcal{C}\left(1 - \frac{3r}{4}; \frac{3r}{4}\right).$$



Proof of the application (lower bound)

It follows from the rephrased main result that

$$P\{\gamma[0,\infty) \cap \mathcal{C}'_r \neq \emptyset\} \asymp r^{4a-1}$$

and so there exists a constant c'_a such that

$$1 - c'_a r^{4a-1} \leq P\{\gamma[0,\infty) \cap \mathcal{C}'_r = \emptyset\}.$$

However, it clearly follows that

$$P\{\gamma[0,\infty) \cap \mathcal{C}'_r = \emptyset\} \leq \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\}$$

where T_z is the swallowing time of the point $z \in \overline{\mathbb{H}}$ (and the infimum is over all $z \in \mathcal{C}_r$ <u>not</u> $z \in \mathcal{C}'_r$). From this we conclude that there exists a constant c'_a such that

$$1 - c'_a r^{4a-1} \leq \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\}.$$

Proof of the application (upper bound)

In order to derive an upper bound, we use a method due to Dubédat.

Let g_t denote the solution to the chordal Loewner equation with driving function $U_t = -B_t$ where B_t is a standard one-dimensional Brownian motion with $B_0 = 0$. For $t < T_1$, the swallowing time of the point 1, consider the conformal transformation $\tilde{g}_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ given by

$$\tilde{g}_t(z) = \frac{g_t(z) + B_t}{g_t(1) + B_t}, \quad \tilde{g}_0(z) = z.$$

Note that $\tilde{g}_t(\gamma(t)) = 0$, $\tilde{g}_t(1) = 1$, $\tilde{g}_t(\infty) = \infty$, and that $\tilde{g}_t(z)$ satisfies the stochastic differential equation

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] \frac{dt}{(g_t(1) + B_t)^2} + [1 - \tilde{g}_t(z)] \frac{dB_t}{g_t(1) + B_t}.$$

If we now perform a time-change and also denoted the time-changed flow by $\{\tilde{g}_t(z),\ t \geq 0\}$, then then $\tilde{g}_t(z)$ satisfies the SDE

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] dt + [1 - \tilde{g}_t(z)]\, dB_t$$

Dubédat showed that for all $\kappa > 0$, this does not explode in finite time (wp1).

Proof of the application (upper bound) (cont.)

Therefore, if F is an analytic function on $\mathbb{H}$ such that $\{F(\tilde{g}_t(z)),\ t \geq 0\}$ is a local martingale, then Itô's formula implies that F must be a solution to the differential equation

$$w(1-w)F''(w) + [2a - (2-2a)w]F'(w) = 0.$$

An explicit solution is given by

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2} d\zeta$$

which is normalized so that $F(0) = 0$ and $F(1) = 1$.

Note that this is a Schwarz-Christoffel transformation of the upper half plane onto the isosceles triangle whose interior angles are $(1-2a)\pi$, $(1-2a)\pi$, and $(4a-1)\pi$.

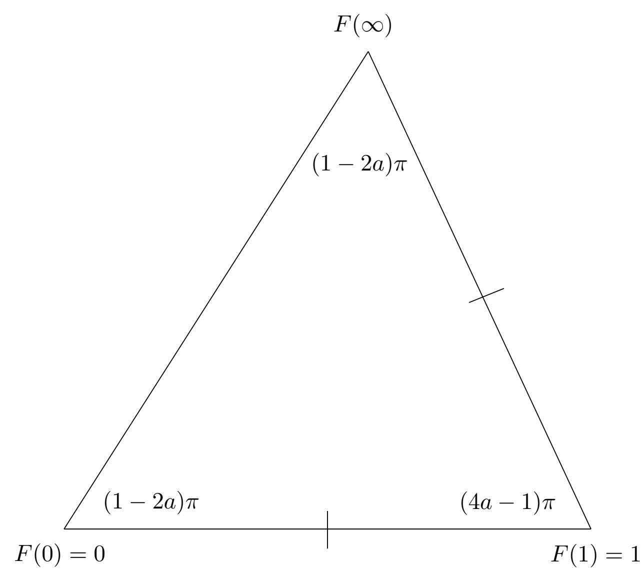
Proof of the application (upper bound) (cont.)

If

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2}\, d\zeta,$$

then the vertices of the traingle are at $F(0) = 0$, $F(1) = 1$, and

$$F(\infty) = \frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(2-4a)\Gamma(4a-1)} e^{(1-2a)\pi i}.$$



Proof of the application (upper bound) (cont.)

Apply the optional sampling theorem to the martingale $F(\tilde{g}_{t \wedge T_z \wedge T_1}(z))$ to find that for $z \in \mathbb{H}$,

$$F(\tilde{g}_0(z)) = F(z) = F(0)P\{T_z < T_1\} + F(1)P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}$$
$$= P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}. \qquad (*)$$

Consequently, identifying the imaginary and real parts of $(*)$ implies that

$$\Re\{F(z)\} = P\{T_z = T_1\} + \Re\{F(\infty)\}P\{T_z > T_1\}.$$

Since $\Re\{F(\infty)\} \geq 0$, we conclude $P\{T_z = T_1\} \leq \Re\{F(z)\} \leq |F(z)|$.

But now integrating along the straight line from 0 to z gives

$$|F(z)| \leq 1 - \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_{|z|}^{1} \rho^{-2a}(1-\rho)^{4a-2} d\rho$$

which relied on the fact that $4a - 2 < 0$.

Proof of the application (upper bound) (cont.)

If $z \in \mathcal{C}_r$ so that $0 < 1 - \frac{3r}{2} \leq |z| \leq 1 - \frac{r}{2} < 1$ by definition, then

$$\int_{|z|}^{1} \rho^{-2a}(1-\rho)^{4a-2} d\rho \geq \frac{2^{1-4a}}{4a-1} r^{4a-1}.$$

Hence,

$$P\{T_z = T_1\} \leq |F(z)| \leq 1 - c_a'' r^{4a-1}$$

where

$$c_a'' = \frac{2^{1-4a}}{4a-1} \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)}.$$

Taking the supremum of the previous expression over all $z \in \mathcal{C}_r$ gives us the required upper bound.