The Green's function for the radial Schramm-Loewner evolution

Michael J. Kozdron

University of Regina

http://stat.math.uregina.ca/∼kozdron/

Third Conference of Tsinghua Sanya International Mathematics Forum
January 8, 2013

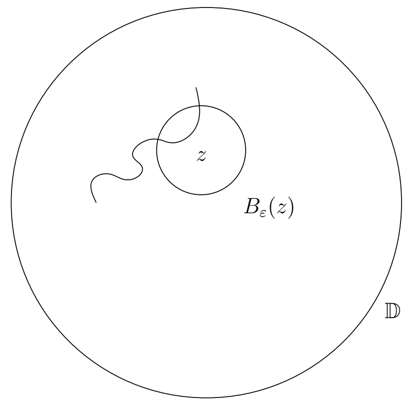
Based on joint work with Tom Alberts (Caltech) and Greg Lawler (Chicago).
J. Phys. A, 45:494015, December 2012.

Thanks to: Mathematical Sciences Research Institute

Heuristic derivation of the Green's function

Consider $\mathbb{D}$, the unit disk centred at 0 in the complex plane.

Suppose that γ is a random curve lying in $\mathbb{D}$.



Consider d, the "fractal dimension" of the curve. By definition,

$$N_\varepsilon \approx \varepsilon^{-d}$$

as $\varepsilon \downarrow 0$ where N_ε is the number of balls of radius ε needed to cover the curve.

(Actually, this is box-counting dimension and provides an upper bound for the Hausdorff dimension.)

Let's try to figure out

$$\mathbf{P}\left\{\gamma[0,\infty) \cap B_\varepsilon(z) \neq \emptyset\right\}.$$

Assume that γ is "equally likely" to pass through any ball covering $\mathbb{D}$. Randomly select a ball of radius ε. This suggests

$$\mathbf{P}\left\{\gamma[0,\infty)\cap B_\varepsilon(z)\neq\emptyset\right\}\approx\frac{\#\text{ of balls needed to cover }\gamma[0,\infty)}{\#\text{ of balls needed to cover }\mathbb{D}}\approx\frac{N_\varepsilon}{\varepsilon^2}\approx\frac{\varepsilon^{-d}}{\varepsilon^2}$$
$$=\varepsilon^{2-d}$$

so that

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{d-2}\mathbf{P}\left\{\gamma[0,\infty)\cap B_\varepsilon(z)\neq\emptyset\right\}\qquad(*)$$

should exist.

If our random curve is a radial SLE_κ path, then $(*)$ should be the Green's function for radial SLE.

A note on terminology

Why do we call this a Green's function?

Recall that the usual Green's function for the Laplacian for $\mathbb{D}$ has a Brownian motion interpretation.

It is the expected number of visits to (a neighbourhood of) z by BM starting at 0 before exiting $\mathbb{D}$ (suitably normalized).

By construction

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P} \left\{ \gamma[0, \infty) \cap B_\varepsilon(z) \neq \emptyset \right\}$$

is the expected spatial density for the radial SLE curve.

A brief history

The question about the Hausdorff dimension of the SLE path was posed very early on in the development of SLE ($\sim$ 2000).

Rohde and Schramm (2005) studied chordal SLE and gave an upper bound for the Hausdorff dimension by basically analyzing the Green's function for chordal SLE.

Beffara (2008) completed the proof of the Hausdorff dimension of the chordal SLE path

$$d = \min\left\{1 + \frac{\kappa}{8}, 2\right\}$$

by rigorously proving a lower bound.

However, the phrase Green's function for SLE was not really used until 2010.

Lawler and Werness (2012) proved the existence of the multi-point Green's function for chordal SLE and gave a new proof of Beffara's estimate.

A brief history

The Green's function for radial SLE has not been previously studied.

Our original motivation comes from joint work in progress with Tom Alberts and Robert Masson. We are trying to prove convergence of loop-erased random walk to radial SLE_2 in the natural parametrization.

Basically, the SLE natural parametrization occupation measure is absolutely continuous with respect to Lebesgue measure and its density is the radial SLE Green's function:

$$\mathbb{E}[\mu(\mathrm{d}z)] = G(z)\,\mathrm{d}z.$$

However, $G(z)$ was not known to exist so we needed to prove its existence.

At the same time, Kang and Makarov were developing a general framework for studying SLE martingale observables using conformal field theory. The chordal and radial SLE Green's functions fit in their framework, but they were unable to find an explicit formula for the radial SLE Green's function.

A technicality

Note that

$$\mathbf{P}\{\gamma[0,\infty) \cap B_\varepsilon(z) \neq \emptyset\} = \mathbf{P}\{\mathrm{dist}(\gamma[0,\infty), z) < \varepsilon\}.$$

However, instead of working with Euclidean distance (which is seemingly more natural), we need to work with conformal distance.

It is not known whether or not

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\{\mathrm{dist}(\gamma[0,\infty), z) < \varepsilon\}$$

exists. Instead we work with

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\{\Upsilon_\infty(z) < \varepsilon\}$$

where

$$\Upsilon_\infty(z) = \lim_{t \to \infty} \Upsilon_t(z)$$

and $\Upsilon_t(z)$ is $1/2$ times the conformal radius of $D \setminus \gamma[0,t]$. Recall that

$$\mathsf{CR}_A(z) = \frac{1}{|f'(z)|}$$

where $f : A \to \mathbb{D}$ with $f(z) = 0$.

Definitions of the chordal and radial SLE Green's functions

By the Koebe $1/4$-theorem,

$$\Upsilon_\infty(z) \asymp \mathrm{dist}(\gamma[0,\infty), z).$$

Chordal SLE ($D = \mathbb{H}$)

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\left\{\Upsilon_\infty(z) < \varepsilon\right\} = c^* \overline{G}_{\mathbb{H}}(z; 0, \infty)$$

Radial SLE ($D = \mathbb{D}$)

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\left\{\Upsilon_\infty(z) < \varepsilon\right\} = c^* G_{\mathbb{D}}(z; 0, 1)$$

The normalizing constant c^* is the same for both radial and chordal.

Existence of the chordal SLE Green's function

In the case of chordal SLE, explicit calculations are possible to show that the Green's function exists. Rohde and Schramm, Beffara, and Lawler have all contributed separately to this statement.

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\left\{\Upsilon_\infty(z) < \varepsilon\right\}$$

exists and equals

$$c^* \overline{G}_{\mathbb{H}}(z; 0, \infty)$$

where

$$\overline{G}_{\mathbb{H}}(z; 0, \infty) = [\operatorname{Im}(z)]^{d-2} \sin^{4a-1}(\arg z)$$

and

$$c^* = 2 \left[\int_0^\pi \sin^{4a} \theta \, \mathrm{d}\theta\right]^{-1}$$

with

$$d = 1 + \frac{\kappa}{8} \quad \text{and} \quad a = \frac{2}{\kappa}.$$

Almost complete derivation of chordal SLE Green's function

Suppose that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \mathbf{P}\left\{\Upsilon_\infty(z) < \varepsilon\right\} = c^* \overline{G}_{\mathbb{H}}(z; 0, \infty).$$

Let

$$\mathcal{F}_t = \sigma(\gamma(s), 0 \le s \le t) \quad \text{and} \quad \rho = \inf\{t : \Upsilon_t(z) = \varepsilon\}.$$

This means

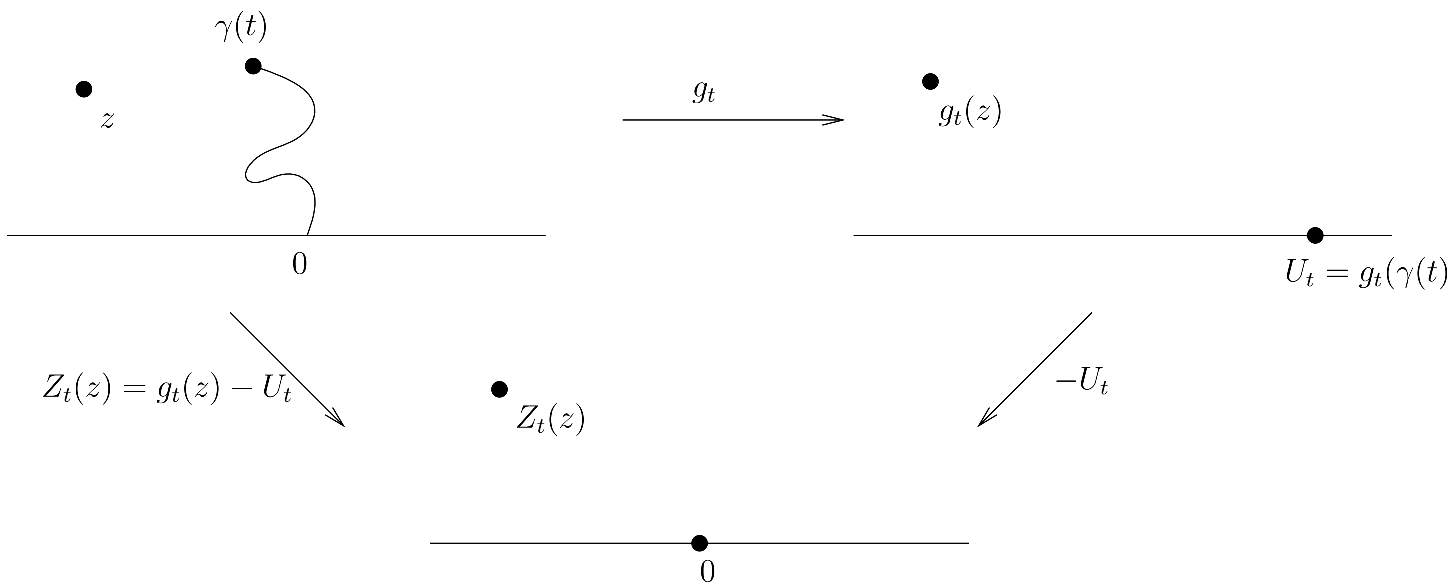
$$N_t = \mathbf{P}\left\{\Upsilon_\infty(z) \le \varepsilon \mid \mathcal{F}_t\right\}$$

should be a local martingale for $0 \le t \le \rho$.

If so,

$$M_t = \mathbb{E}[\overline{G}_{\mathbb{H}}(z; 0, \infty) | \mathcal{F}_t]$$

should be a local martingale.



By the domain Markov property of SLE and assumed conformal covariance of Green's function,

$$
\begin{aligned}
M_t = \mathbb{E}[\overline{G}_{\mathbb{H}}(z; 0, \infty) | \mathcal{F}_t] &= \overline{G}_{\mathbb{H}\setminus\gamma[0,t]}(z; \gamma(t), \infty) \\
&= |g_t'(z)|^{2-d}\overline{G}_{\mathbb{H}}(g_t(z); U_t, \infty) \\
&= |g_t'(z)|^{2-d}\overline{G}_{\mathbb{H}}(Z_t(z); 0, \infty).
\end{aligned}
$$

Hence, we have a local martingale $M_t = |g_t'(z)|^{2-d}\overline{G}(Z_t)$.

Observe that

$$\mathrm{d}Z_t = \mathrm{d}g_t(z) - \mathrm{d}U_t = \frac{2}{Z_t}\,\mathrm{d}t - \mathrm{d}U_t.$$

Using Itô's formula on M_t gives

$$\mathrm{d}M_t = \boxed{}\,\mathrm{d}U_t + \boxed{}\,\mathrm{d}t.$$

Let $z = (x, y)$. Since M_t is a local martingale, we must have the coefficient for $\mathrm{d}t$ equal to 0. This gives a PDE for $\overline{G}$:

$$\frac{1}{2}H_{xx}(x,y) - \frac{ay}{x^2+y^2}H_y(x,y) + \frac{ax}{x^2+y^2}H_x(x,y) + \frac{(4a-1)y^2}{2(x^2+y^2)^2}H(x,y) = 0$$

where $H(x,y) = y^{2-d}\overline{G}(x,y)$.

Chordal SLE scaling implies $H(x,y) = \phi(y/x)$ so we find an ODE for ϕ that can be solved explicitly.

$$\overline{G}(x,y) = y^{d-2}\left(\frac{y}{\sqrt{x^2+y^2}}\right)^{4a-1}.$$

Similar calculations for the radial SLE Green's function

Suppose that $\gamma : [0, \infty) \to \mathbb{D} \setminus \{0\}$ is a radial SLE$_\kappa$ and that g_t are the usual conformal transformations with $g_t(\gamma(t)) = e^{i2B_t}$. If $z \in \mathbb{D}$, then $g_t(z)$ satisfies the differential equation

$$\partial_t g_t(z) = 2a g_t(z) \frac{e^{i2B_t} + g_t(z)}{e^{i2B_t} - g_t(z)}, \quad g_0(z) = z.$$

Let

$$Z_t(z) = e^{-2iB_t} g_t(z)$$

and suppose we try to do the same thing for radial SLE. This suggest that

$$M_t = |g_t'(z)|^{2-d} G_{\mathbb{D}}(Z_t(z); 0, 1)$$

is a local martingale. Using Itô's formula on M_t gives

$$\mathrm{d}M_t = \boxed{}\, \mathrm{d}B_t + \boxed{}\, \mathrm{d}t.$$

Since M_t is a local martingale, we must have the coefficient for $\mathrm{d}t$ equal to 0.

The PDE for the radial SLE Green's function

When working with chordal SLE in the upper half plane, the most natural coordinates are cartesian: $z = x + iy$. Chordal scaling (i.e., conformal invariance) suggests that chordal SLE martingales should be functions of the ratio y/x. There are numerous examples of chordal PDEs that reduce to ODEs and yield exact solutions: Cardy's formula, Schramm's left passage probability, Fomin's identity for SLE_2, multiple SLE, etc.

Working with radial SLE in the unit disk $\mathbb{D}$ suggests that the most natural coordinates are polar: $z = re^{i\theta}$. Unfortunately there is no obvious radial scaling that reduces PDEs to ODEs. As such, most studies with radial SLE avoid the martingale-to-PDE approach. Here is one example of a radial PDE that yields an exact solution: the Green's function for radial SLE_4.

The PDE for the radial SLE Green's function

Working in polar coordinates $z = re^{i\theta}$ and doing a lot of calculations implies

$$0 = H_{\theta\theta} + \frac{2ar\sin\theta}{1+r^2-2r\cos\theta}H_\theta + \frac{ar(1-r^2)}{1+r^2-2r\cos\theta}H_r + \left(a-\frac{1}{4}\right)\left(\frac{\partial}{\partial\theta}\frac{2r\sin\theta}{1+r^2-2r\cos\theta}\right)H.$$

where

$$H(r,\theta) = r^{2-d}G_{\mathbb{D}}(re^{i\theta};0,1)$$

Remark. It took some time to determine that this was the cleanest formulation of the PDE.

Remark. Notice that the Poisson kernel and its complex conjugate appear as coefficients.

An explicit solution when $\kappa = 4$

A natural guess for the solution is

$$H(r,\theta) = \left(\frac{1-r^2}{1+r^2-2r\cos\theta}\right)^{\zeta}$$

for some ζ.

After substituting, we find that the PDE is satisfied for this guess iff $\zeta = a = 1/2$. Note that $a = 2/\kappa$ so

$$a = \frac{1}{2} \quad \text{iff} \quad \kappa = 4.$$

Remark. There are other natural guesses for the solution to the PDE. However, none of them actually produces a solution.

An explicit solution when $\kappa = 4$

The Green's function for radial SLE_4 from 1 to 0 in $\mathbb{D}$ is

$$G_{\mathbb{D}}(re^{i\theta}; 0, 1) = r^{-1/2} \left(\frac{1 - r^2}{1 + r^2 - 2r\cos\theta} \right)^{1/2}$$

or, equivalently,

$$G_{\mathbb{D}}(z; 0, 1) = \sqrt{\frac{1 - |z|^2}{|z| \cdot |1 - z|^2}}$$

where $z = re^{i\theta} \in \mathbb{D}$.

$\mathbb{D}$ *may not always be the most natural radial SLE domain*

This has not really been emphasized much in the radial SLE literature, but the covering space of $\mathbb{D}$ is a useful canonical domain for analyzing radial SLE.

Let $\mathbb{H}^*$ denotes the upper half plane modulo π.

$\mathbb{H}^*$ is the covering space of $\mathbb{D}$ and can be identified with a half-infinite cylinder of circumference π.

If $z = x + iy$ with $\mathrm{Re}(z) \in [-\pi/2, \pi/2)$, then

$$\frac{i}{\pi}\cot z = \frac{1}{\pi}\frac{\sinh y \cosh y}{|\sin z|^2} + \frac{i}{\pi}\frac{\sin x \cos x}{|\sin z|^2}.$$

The real part is the Poisson kernel for $\mathbb{H}^*$.

$$u(z) = u(x,y) = \frac{\sinh y \cosh y}{\sin^2 x + \sinh^2 y}, \qquad v(z) = v(x,y) = \frac{\sin x \cos x}{\sin^2 x + \sinh^2 y}.$$

$$-v_x(z) = \mathrm{Re}(\csc^2 z) = \frac{\sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y}{(\sin^2 x + \sinh^2 y)^2}.$$

Lemma. If $p = (4a-1) - 2(2-d)$, $\zeta = (4a-1) - (2-d)$, and

$$H(z) = |\sin z|^p \, u(z)^\zeta,$$

then H satisfies the differential equation

$$\frac{1}{2} H_{xx}(z) + av(z) H_x(z) - au(z) H_y(z) - \left(\frac{1}{4} - a\right) v_x(z) H(z) + apH(z) = 0.$$

In particular,

$$N_t = N_t(z) = e^{apt} |g_t'(z)|^{2-d} H(Z_t(z))$$

is a local martingale.

Chordal in $\mathbb{H}$

$$\overline{G}(x,y) = y^{d-2} H(x,y)$$

$$\frac{1}{2} H_{xx}(z) - \frac{ay}{x^2+y^2} H_y(z) + \frac{ax}{x^2+y^2} H_x(z) + \frac{(4a-1)y^2}{2(x^2+y^2)^2} H(z) = 0$$

$$M_t = |g_t'(z)|^{2-d} \overline{G}_{\mathbb{H}}(Z_t(z))$$

Radial in $\mathbb{H}^*$

$$\frac{1}{2} H_{xx}(z) - au(z) H_y(z) + av(z) H_x(z) + \frac{4a-1}{4} v_x(z) H(z) + apH(z) = 0.$$

$$N_t = e^{apt} |g_t'(z)|^{2-d} H(Z_t(z))$$

The main theorem

Suppose that $p = (4a - 1) - (2 - d)$, $\zeta = (4a - 1) - 2(2 - d)$, and

$$H(z) = |\sin z|^p u(z)^\zeta.$$

The Green's function for radial SLE in $\mathbb{H}^*$ is

$$G(z) = H(z)\,\Phi(z),$$

where

$$\Phi(z) = \mathbb{E}^*[e^{-apT}].$$

In other words, if $z \in \mathbb{H}^*$, then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{d-2}\,\mathbf{P}\{\Upsilon_\infty(z) \le \varepsilon\} = c^*\,G(z) \quad \text{where} \quad c^* = 2\left[\int_0^\pi \sin^{8/\kappa}\theta\,\mathrm{d}\theta\right]^{-1}.$$

If $\kappa = 4$, then $G(z) = H(z)$.

$$\boxed{\Phi(z) = \mathbb{E}^*[e^{-apT}]}$$

$\mathbb{E}^*$ denotes expectation with respect to the SLE measure weighted by the local martingale N_t, i.e., an expectation with respect to SLE conditioned to go through z.

SLE: originally driven by Brownian motion B_t.

weighted SLE: use Girsanov to weight by local martingale N_t

$$\mathrm{d}N_t = J_t N_t \,\mathrm{d}B_t \quad \text{so} \quad \mathrm{d}B_t = J_t \,\mathrm{d}t + \mathrm{d}W_t$$

where W_t is a standard BM wrt new measure

$Z_t(z)$: map that sends tip of the curve to the origin

$T = \inf\{Z_t(z) = 0\}$

Elements of proof of the theorem

- A very careful comparison of chordal SLE in $\mathbb{H}$ with radial SLE in $\mathbb{H}^*$ in disk of radius $r \ll 1$ centred at 0. For small times, these processes are close.

- One can compare chordal SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$ and radial SLE from 0 to i in $\mathbb{H}$ by tilting by a particular local martingale.

- Radial SLE$_\kappa$ in $\mathbb{H}^*$ can be obtained from radial SLE$_\kappa$ in $\mathbb{H}$ from 0 to i by the (multiple valued) transformation

$$f(z) = \psi^{-1} \circ \phi(z) = \frac{1}{2i} \log \left[\frac{z-i}{z+i} \right]$$

  where $\psi(z) = e^{2iz}$ which is a conformal transformation of $\mathbb{H}^*$ onto $\mathbb{D} \setminus \{0\}$ and $\phi(z) = (z-i)/(z+i)$ which is a conformal transformation of $\mathbb{H}$ onto $\mathbb{D}$.

- In both cases, one sees that the driving function changes from a standard Brownian motion to one with a drift. Under our conditions, the drift is uniformly bounded, and since time is bounded by $O(r^{1/2})$, we can bound the Radon-Nikodým derivative in the change of measure.

To do

Prove loop-erased random walk converges to SLE_2 in the natural parametrization.