A Random Look at the Schramm-Loewner Evolution

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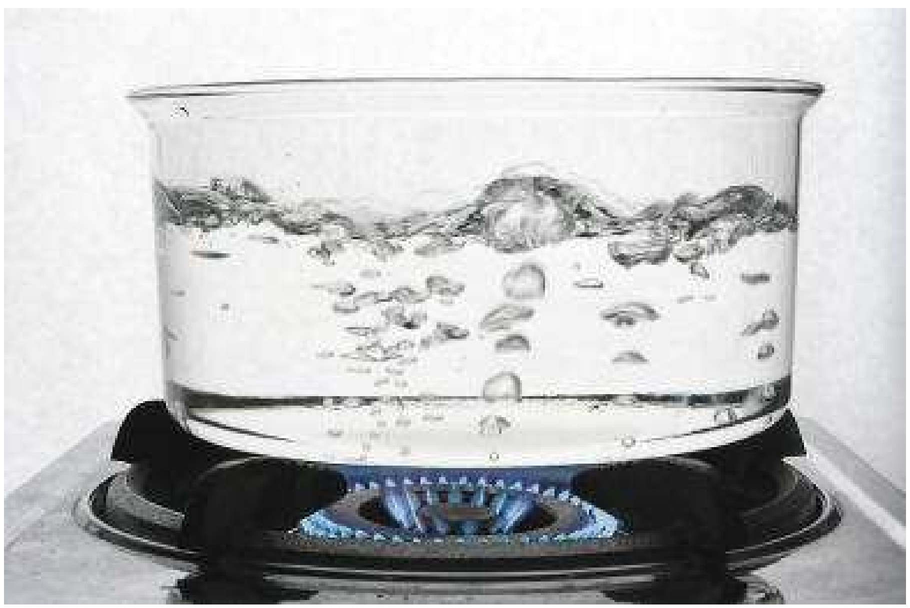
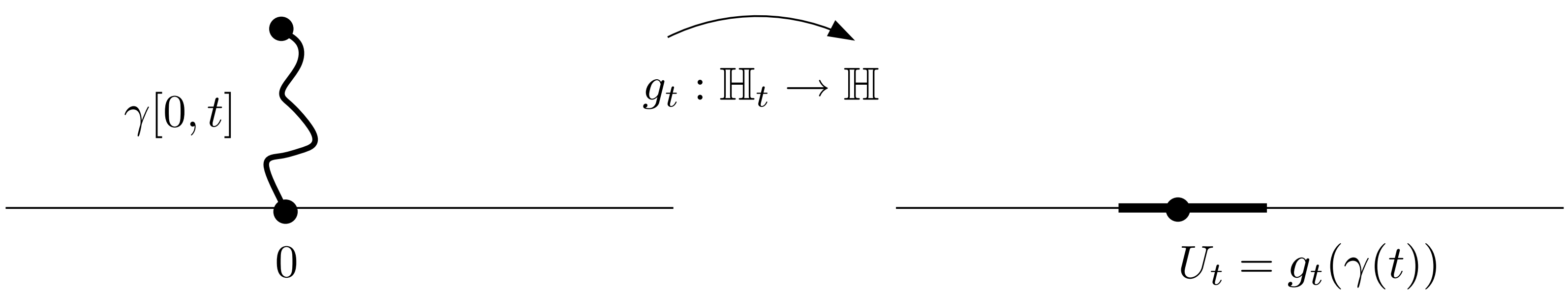
Some Statistical Mechanics



Some Models From Statistical Mechanics

- The self-avoiding walk is a model of polymer chains introduced in 1949 by the Nobel Prize-winning chemist Paul Flory.
- The Ising model was invented by Wilhelm Lenz in 1920 and studied by his student Ernst Ising as a model of ferromagnetism; more generally, it is a simple model of an interacting system.
- The loop-erased random walk is a mathematical model introduced in 1980 by Greg Lawler in an attempt to understand the self-avoiding walk. It was eventually proved to be in a different universality class than the SAW, but is an interesting model in its own right. In the mid-90s it was proved to be intimately connected with the generation of uniform spanning trees.
- Percolation is a model of fluid flow through a porous medium introduced by Simon Broadbent and John Hammersley in 1957.

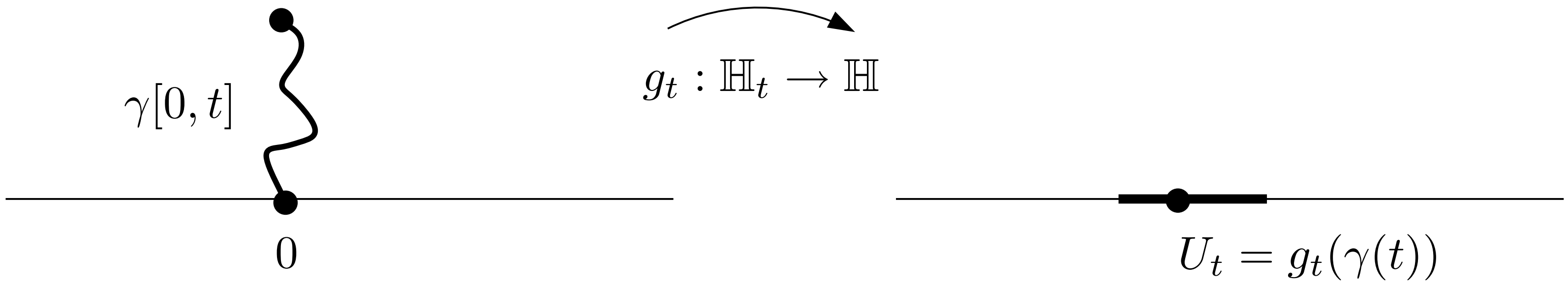
The Picture That Says It All!



- The simple curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- The curve γ never re-visits $\mathbb{R}$; that is, $\gamma(0,t) \subset \mathbb{H}$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0,t]$ denotes the slit plane.
- $g_t : \mathbb{H}_t \to \mathbb{H}$ is a conformal map;
- $U_t := g_t(\gamma(t))$ is the unique point on $\mathbb{R}$ that is the image of the tip, $\gamma(t)$.
- $t \mapsto U_t$ is continuous.

What is SLE?

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by the Loewner equation.



We (uniquely) normalize g_t and (re-)parametrize γ in such a way that as $z \to \infty$,

$$g_t(z) = z + \frac{2t}{z} + O\left(|z|^{-2}\right).$$

Theorem. (Loewner 1923)
If $z \in \mathbb{H}$ with $z \notin \gamma[0,\infty]$, then the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the IVP

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

The natural thing to do is to start with a continuous function $t \mapsto U_t$ and solve the Loewner equation.

Solving the Loewner equation gives g_t which conformally maps $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$ and $K_t = \gamma(0, t]$.

While studying loop-erased random walk, Schramm's idea was to let U_t be a Brownian motion! (In retrospect, it is natural.)

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion.

Stochastic Loewner Evolution (aka Schramm-Loewner Evolution)

Definition. SLE$_\kappa$ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

It is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. A deep theorem due to Rohde and Schramm proves this is true.

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$.

SLE$_\kappa$ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

What does SLE look like?

Theorem. (Rohde-Schramm 2001; Lawler-Schramm-Werner 2004)

With probability one,

- $0 < \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding $\mathbb{R}$.

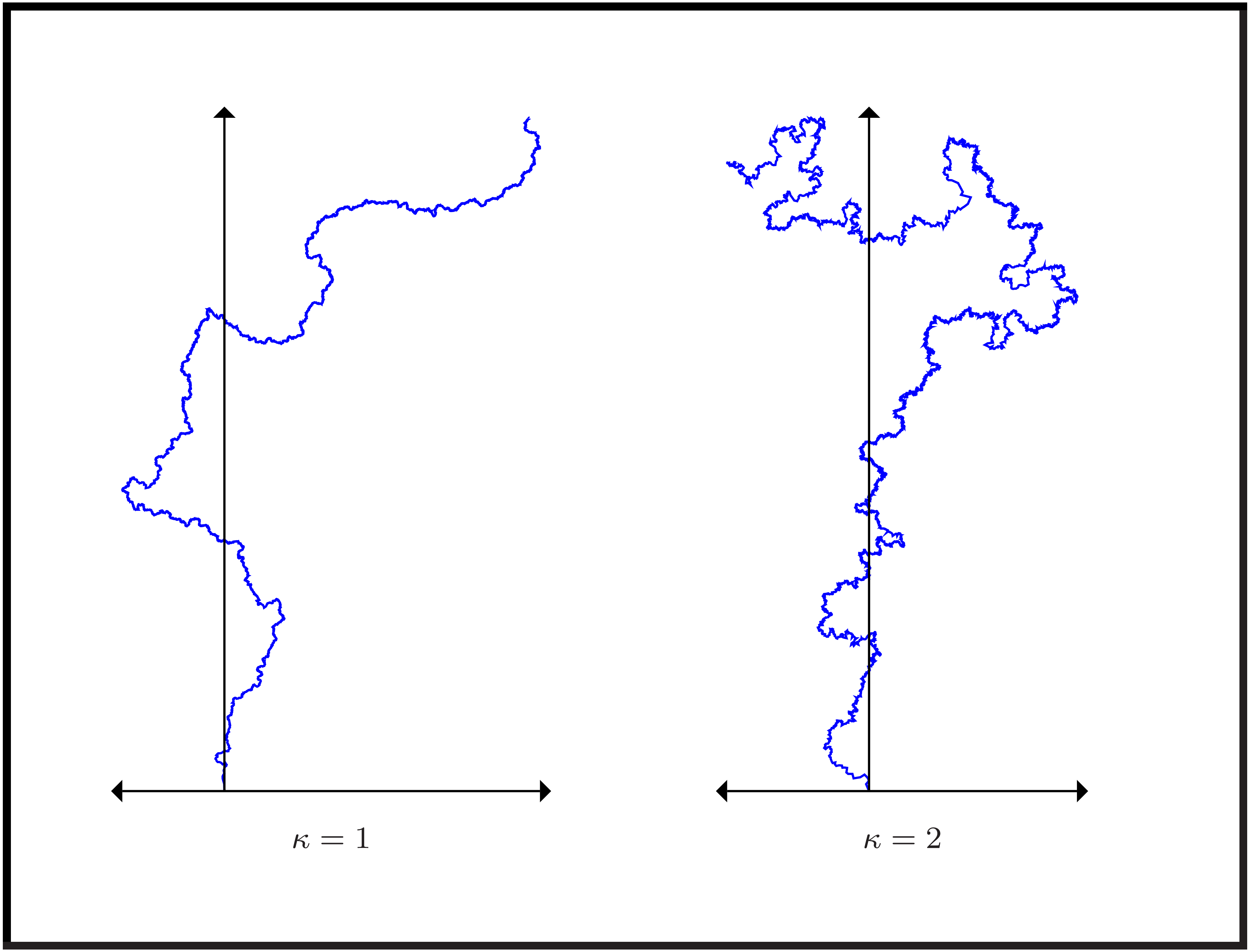
- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit $\mathbb{R}$.

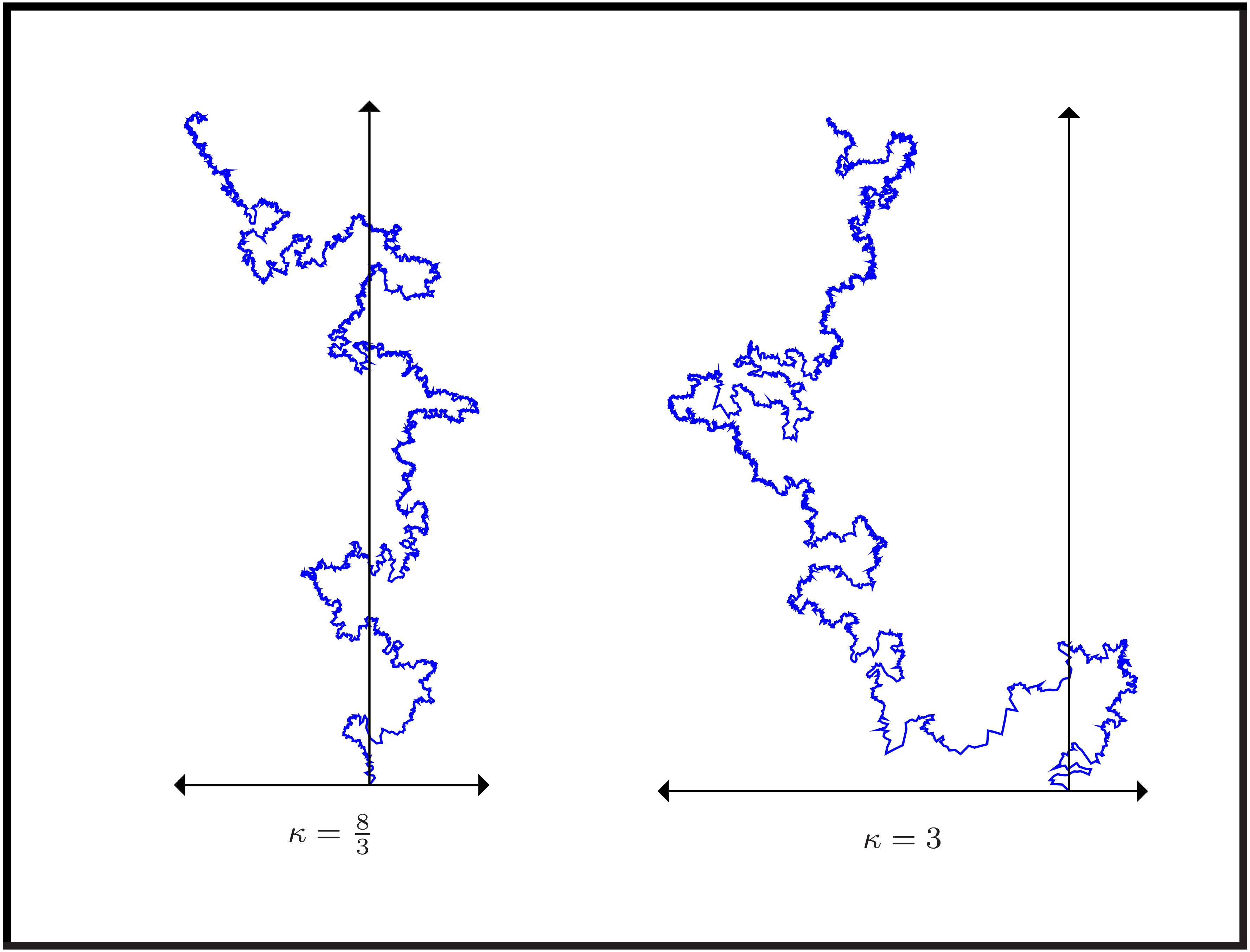
- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Theorem. (Beffara 2004, 2008)

With probability one, the Hausdorff dimension of the SLE$_\kappa$ trace is

$$\min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$


$\kappa = 1$
$\kappa = 2$


$\kappa = \frac{8}{3}$
$\kappa = 3$

Chordal SLE in D

Technically, we have defined **chordal SLE**. That is, SLE connecting two distinct boundary points of a simply connected domain.

Another process known as **radial SLE** connects a boundary point with an interior point.

Schramm originally defined chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. We've outlined his construction.

He then defined chordal SLE$_\kappa$ in D from z to w to be the image of SLE$_\kappa$ in $\mathbb{H}$ under a conformal transformation taking $0 \mapsto z$ and $\infty \mapsto w$.

Everything is defined up to time reparametrization.

There are other constructions of chordal SLE in D. The original way could be described as the "infinitesmal approach" and uses a particular SDE. Another way is to construct a finite measure on curves via martingales and a particular Radon-Nikodym derivative.

Either way, SLE is conformally invariant.

Proving Convergence Results with SLE

The following martingale principle is essentially due to Smirnov and is one way to prove convergence to SLE. (In fact, it is the only known way.)

Martingale Observable Principle. If γ is a random curve that admits a non-trivial conformal martingale

$$M_t(x) = M(x; D, \gamma(t), z, w),$$

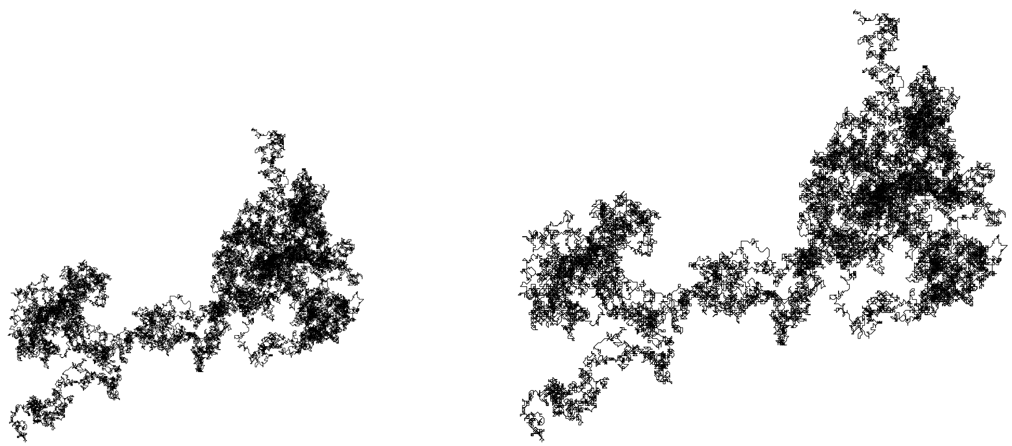
then γ is given by SLE with parameter κ derived from M_t.

- For each model, one needs to find a suitable martingale observable.
- At the moment, there is no general "template" for constructing such martingales.
- Even if one has a martingale observable, there are still technical questions particular to each model about what to do with it and how to actually carry through the proof of convergence.
- In many cases, the martingale observable at time $t \to \infty$ reduces to the probability of a particular event. Computing that probability is often a key step in the proofs of convergence.

The Conformal Invariance Prediction

In 1994, Aizenman, Langlands, Pouliot, and Saint-Aubin conjectured, roughly, that if Λ is a planar lattice with suitable symmetry, and we perform critical percolation on Λ, then as the lattice spacing tends to 0, certain limiting probabilities are invariant under conformal transformations.

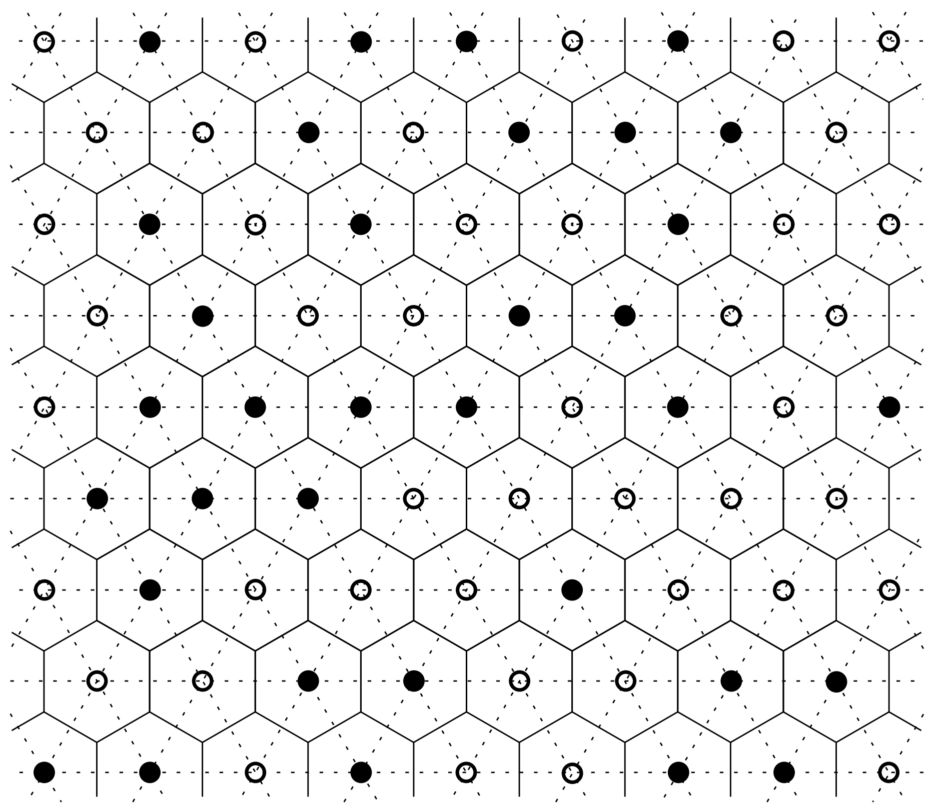
There is a crude analogy to simple random walk here. Simple random walk on any suitable lattice converges to Brownian motion.



The prediction has only been proved for site percolation on the triangular lattice.

Example: Site Percolation on the Triangular Lattice

Site percolation on the triangular lattice can be identified with "face percolation" on the hexagonal lattice (which is dual to the triangular lattice).



The Discrete Percolation Exploration Path

Consider a simply connected, bounded hexagonal domain with two distinguished external vertices x and y.

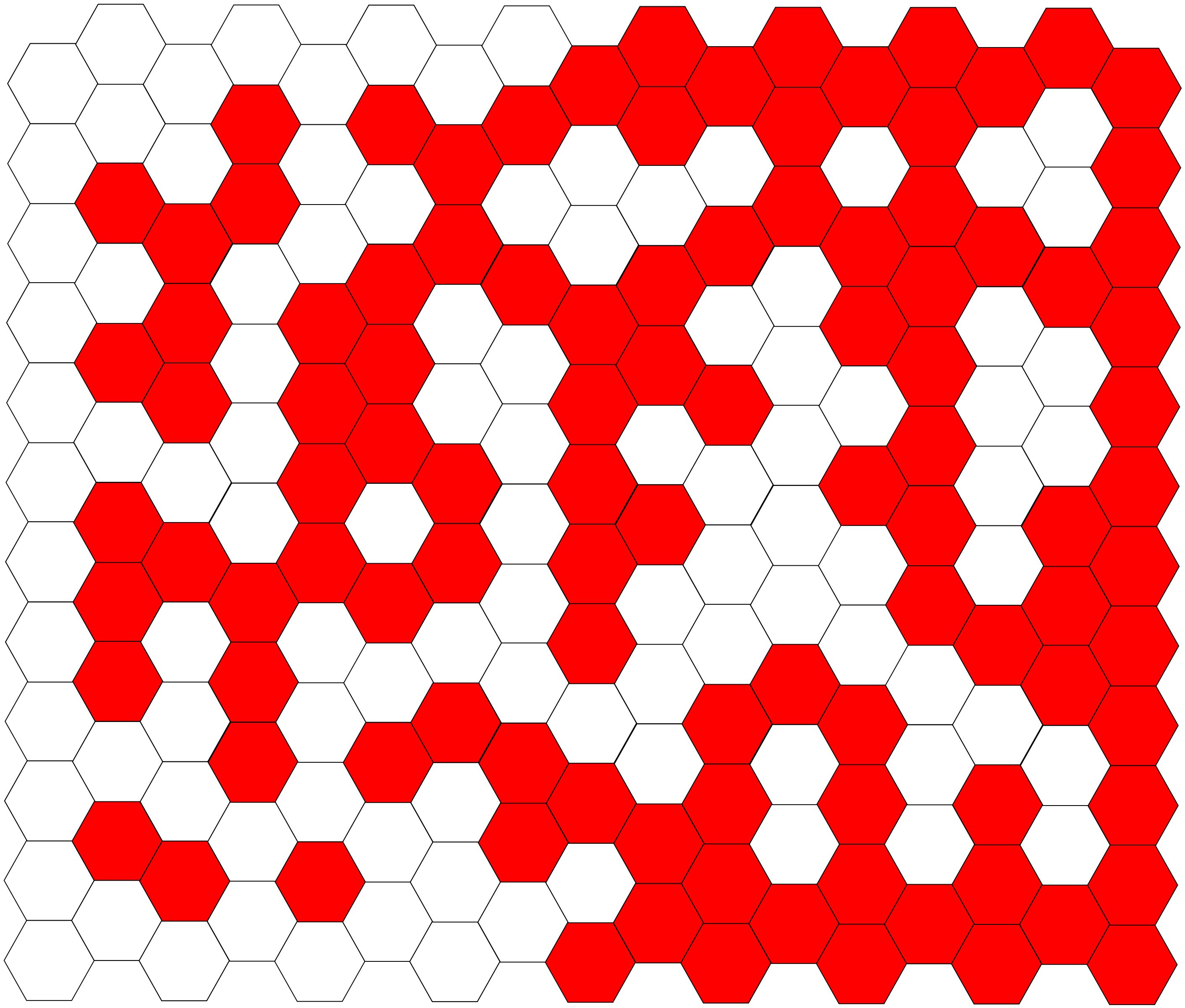
Colour all the hexagons on one half of the boundary from x to y white, and colour all the hexagons on the other half of the boundary from y to x red.

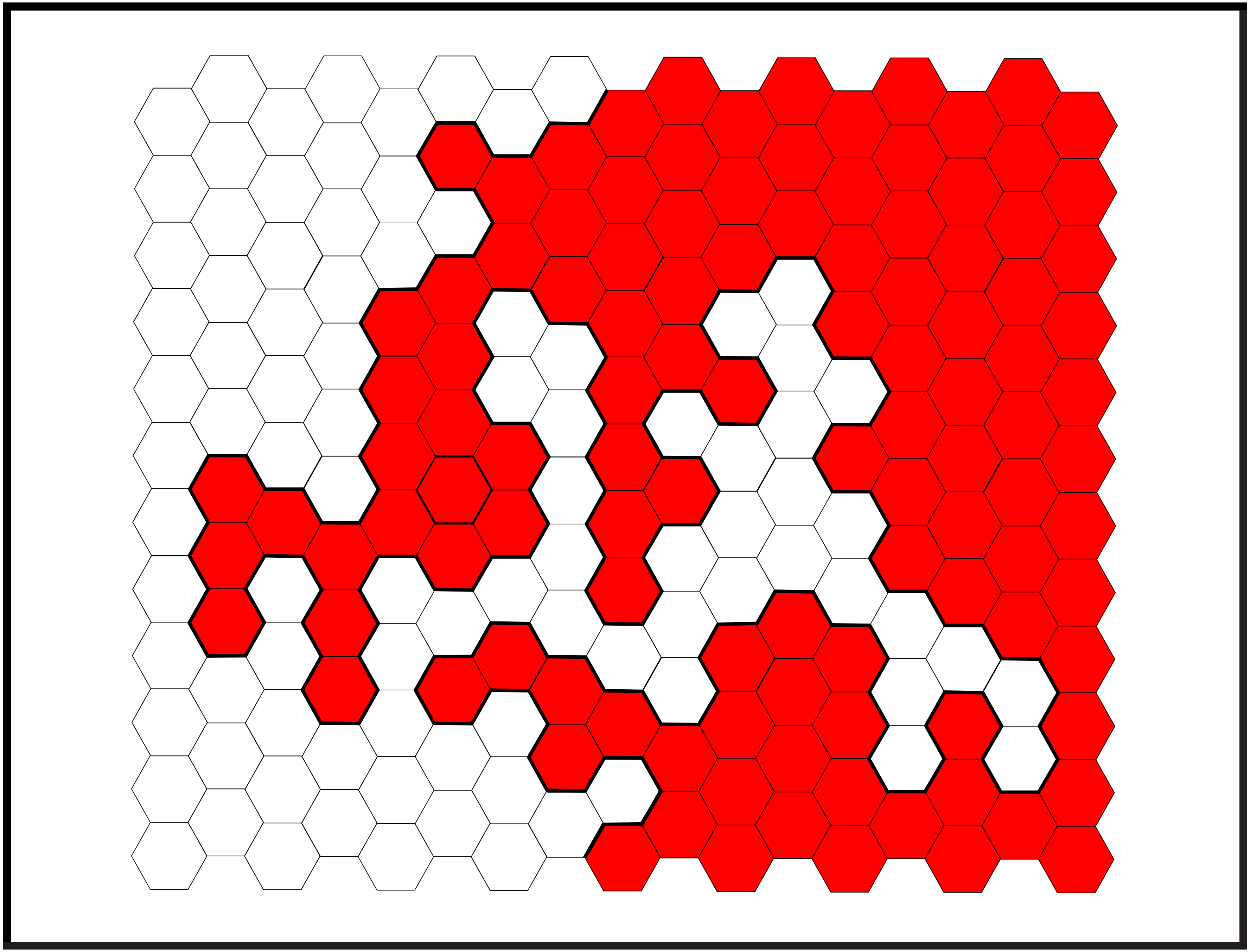
For all remaining interior hexagons colour each hexagon either red or white independently of the others each with probability $1/2$ (i.e., perform critical site percolation on the triangular lattice).

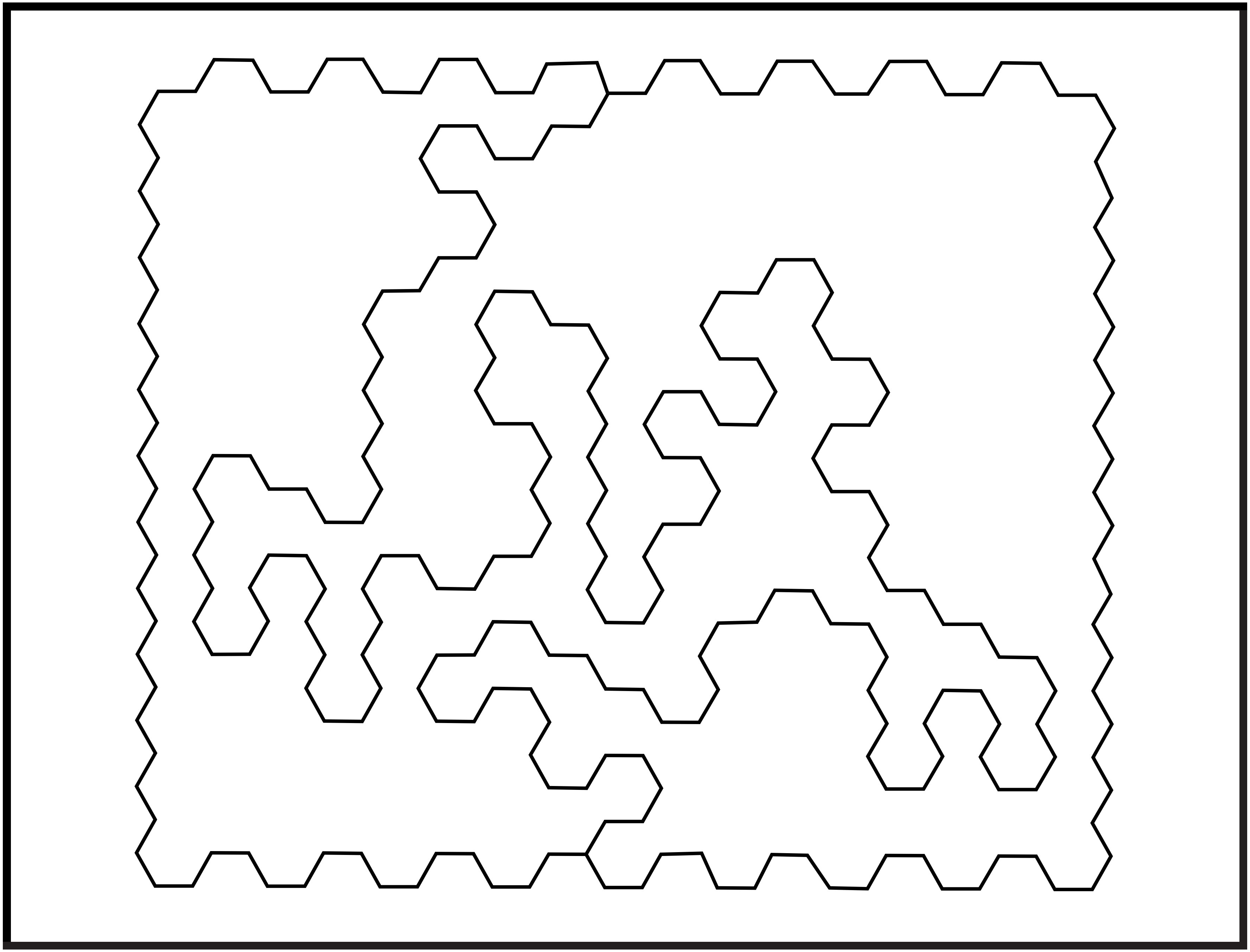
There will be an interface separating the red cluster from the white cluster.

One way is to draw the interface always keeping a red hexagon on the right and a white hexagon on the left.

Another way to visualize the interface is to swallow any islands so that the domain is partitioned into two connected sets.





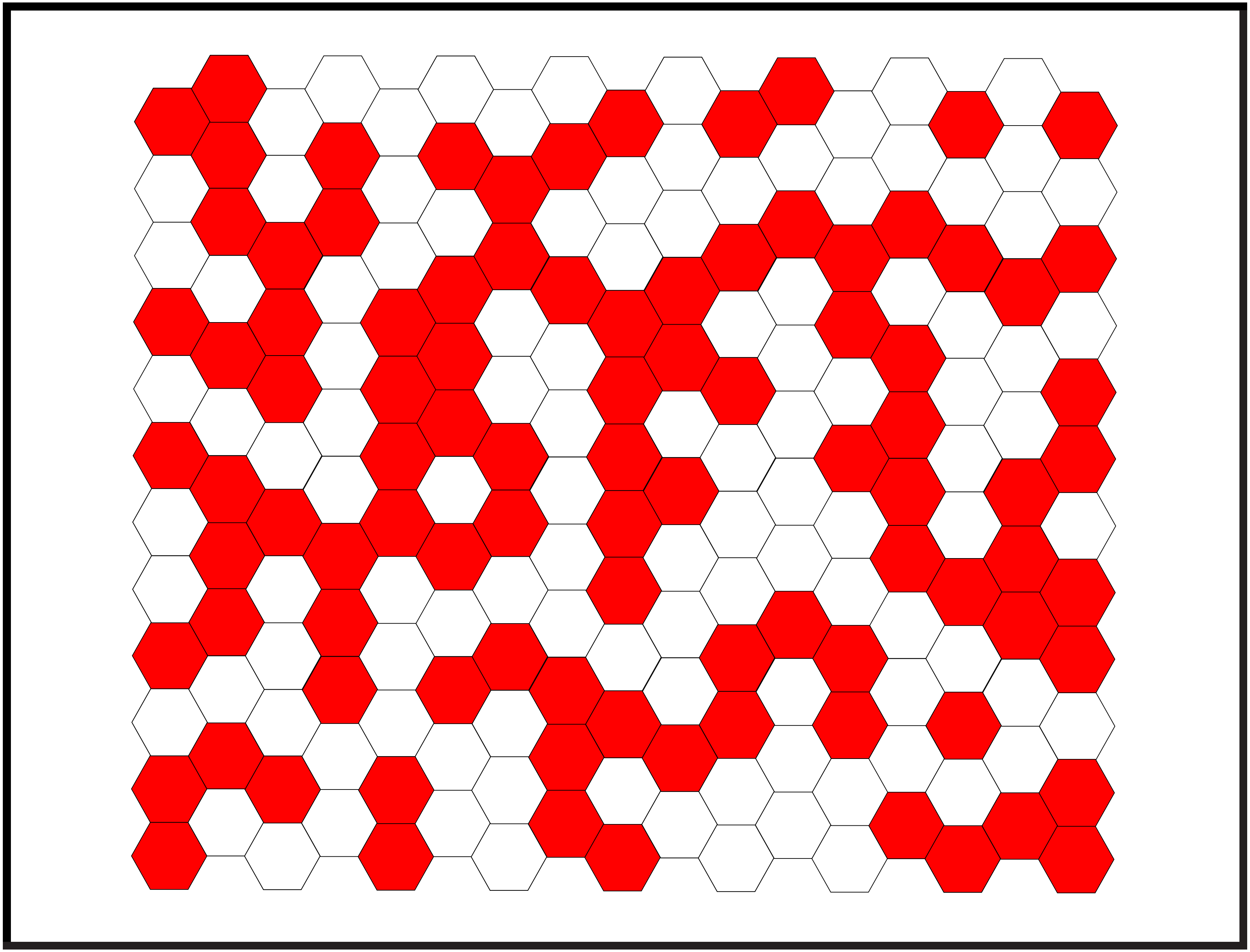


Crossing Probabilities for the Discrete Domain

Consider a simply connected, bounded hexagonal domain D with four distinguished external vertices z_1, z_2, z_3, z_4 ordered counterclockwise. This divides the boundary into four arcs, say A_1, A_2, A_3, A_4.

For all hexagons in D colour each hexagon either red or white independently of the others each with probability 1/2 (i.e., perform critical site percolation on the triangular lattice).

There will necessarily be either a red (open) crossing from A_1 to A_3 or a white crossing from A_2 to A_4.



Approximating the Continuous

Suppose that $D \subset \mathbb{C}$ is a simply connected, bounded Jordan domain containing the origin, and let z_1, z_2, z_3, z_4 be four points ordered counterclockwise around ∂D.

This divides ∂D into 4 arcs, say A_1, A_2, A_3, A_4.

Overlay a suitable lattice with spacing δ over D and consider the resulting lattice-domain D^δ. Identity the original arcs with lattice-domain arcs $A_1^\delta, A_2^\delta, A_3^\delta, A_4^\delta$.

Perform critical percolation on D^δ.

Goal: To understand what happens as $\delta \downarrow 0$?

Question 1: What is the probability that there is a red crossing from A_1^δ to A_3^δ? Call this $P(D;\delta) = P(D, z_1, z_2, z_3, z_4; \delta)$.

Question 2: What is the law or distribution of the scaling limit of the discrete interface?

John Cardy's Formula

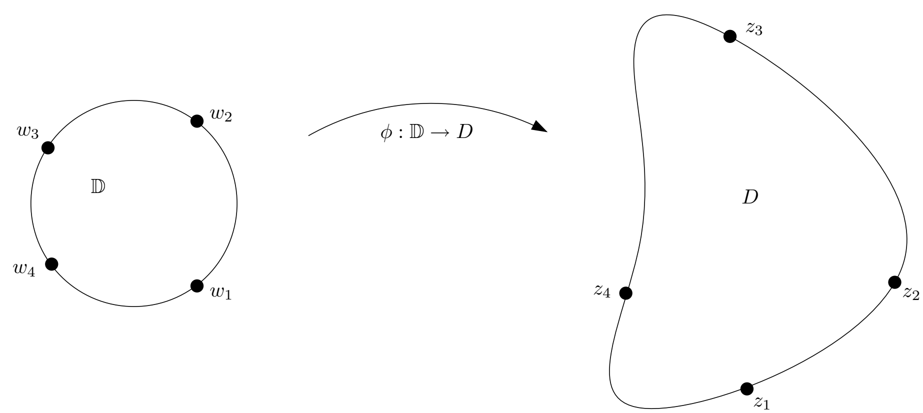
Cardy's Prediction/Formula (1992):

$$\lim_{\delta \to 0} P(D;\delta) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta)$$

where ${}_2F_1$ is the hypergeometric function and

$$\eta = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}$$

is the cross-ratio with $w_j = \phi(z_j)$ where $\phi : \mathbb{D} \to D$ is the unique conformal transformation with $\phi(0) = 0$, $\phi'(0) > 0$.



Lennart Carleson's Observation

Using properties of the hypergeometric function one can write

$$\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} z^{1/3} {}_2F_1(1/3, 2/3; 4/3; z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

Furthermore, the function

$$z \mapsto \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z w^{-2/3}(1-w)^{-2/3}\, dw$$

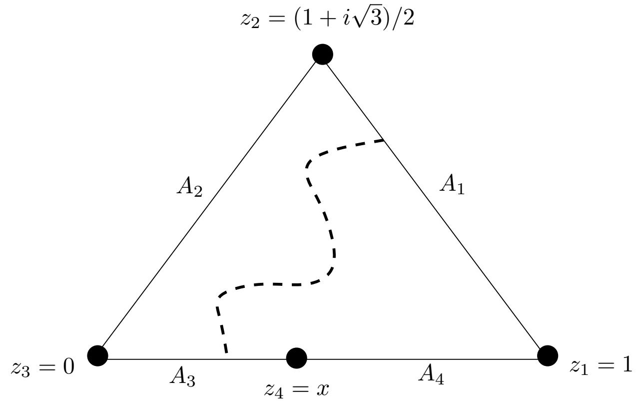
is the Schwarz-Christoffel transformation of the upper half plane onto the equilateral traingle with vertices at 0, 1, and $(1+i\sqrt{3})/2$.

Lennart Carleson's Observation

Hence, if D is this equilateral triangle, then Cardy's prediction takes the particularly nice form

$$\lim_{\delta \to 0} P(D;\delta) = x \tag{$*$}$$

where x is the following:



Theorem: (Smirnov 2001) Cardy's prediction holds for critical site percolation on the trianglular lattice. Smirnov proved $(*)$ and conformal invariance gave it for all Jordan domains D.

The Scaling Limit of the Exploration Process

Thanks to the work of Smirnov and Werner, there is now a precise description of the scaling limit of the interface (i.e., the exploration process).

Suppose that (D, z, w) is a Jordan domain with distinguished boundary points z and w.

Let $(D^\delta, z^\delta, w^\delta)$ be a sequence of hexagonal lattice-domains with spacing δ which approximate (D, z, w).

(Technically, $(D^\delta, z^\delta, w^\delta)$ converges in the Carathéodory sense to (D, z, w) as $\delta \downarrow 0$.)

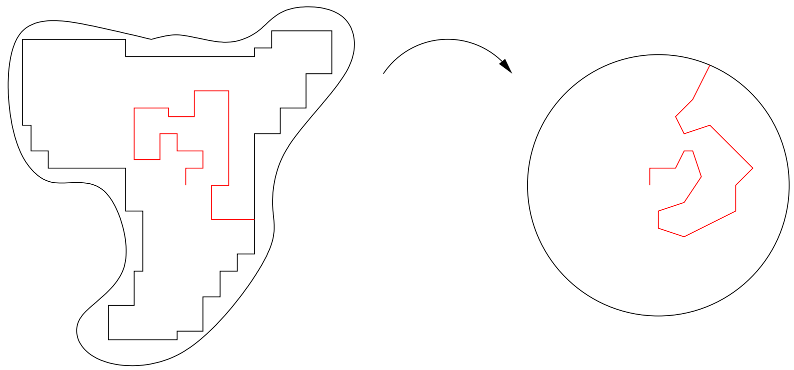
Let $\gamma^\delta = \gamma^\delta(D^\delta, z^\delta, w^\delta)$ denote the spacing δ exploration path.

As $\delta \downarrow 0$, the sequence γ^δ converges in distribution to SLE_6 in D from z to w.

Summary of Convergence Results

- Loop-erased random walk converges to SLE_2 (Lawler, Schramm, Werner)
- Self-avoiding walk should converge to $\mathrm{SLE}_{8/3}$ (Lawler, Schramm, Werner)
- Interfaces in the Ising model converge to SLE_3 (Smirnov)
- Level lines of the discrete Gaussian free field converge to SLE_4 (Schramm, Sheffield)
- Percolation exploration path converges to SLE_6 (Smirnov)
- UST Peano curve converges to SLE_8 (Lawler, Schramm, Werner)

A Rate of Convergence of LERW to SLE(2)



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$. A grid domain is a domain whose boundary is a union of edges of the scaled lattice. That is, D_n is the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting ∂D. Note that D_n is simply connected. Write $V = V(D_n)$ for the set of vertices contained in D_n
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parameterize by capacity.
- $W_n(t) = W_0 e^{i\vartheta_n(t)}$: the Loewner driving function for $\tilde{\gamma}_n$.

A Rate of Convergence of LERW to SLE(2)

Theorem (Beneš-Johansson Viklund-K, 2011). Let $0 < \epsilon < 1/24$ be fixed, and let D be a simply connected domain with $\text{inrad}(D) = 1$. For every $T > 0$ there exists an $n_0 < \infty$ depending only on T such that whenever $n > n_0$ there is a coupling of γ_n with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that

$$\mathbf{P}\left(\sup_{0\leq t\leq T} |W_n(t) - e^{iB(2t)}| > n^{-(1/24-\epsilon)}\right) < n^{-(1/24-\epsilon)}.$$

Recall that

$$W_n(t) = W_n(0)e^{i\vartheta_n(t)}, \quad t \geq 0,$$

denotes the Loewner driving function for the curve $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ parameterized by capacity.

Areas of Active Research

SLE describes the scaling limit of a single interface. What about multiple interfaces? This has been considered from a physical point of view by Bauer, Bernard, and Kytölä (2005). Mathematical approaches have been considered by Dubédat (2006) and by K. and Lawler (2006). Rigorously constructing a measure on multiple non-crossing SLE paths for $4 < \kappa < 8$ is still an open problem. Proving convergence for multiple interfaces in discrete models to multiple SLE is still an open problem.

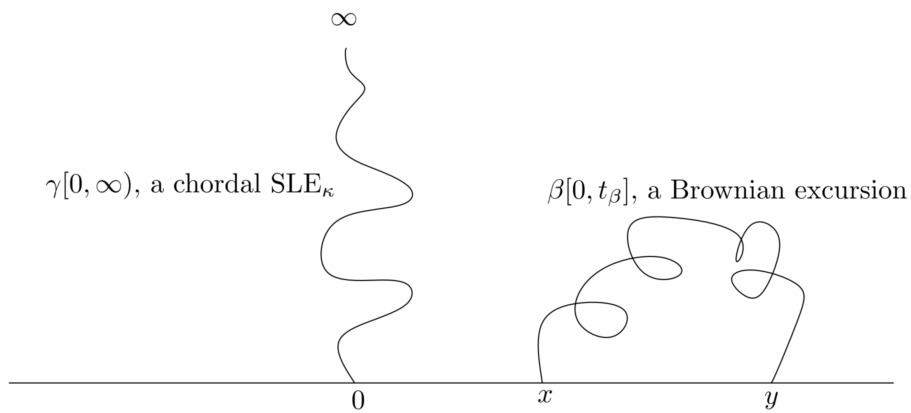
Viewed as a mathematical object, there is interest in distributional properties of the SLE path. Beffara established the Hausdorff dimension of the curve (2004, 2008). Sheffield and Alberts (2008) determined the Hausdorff dimension of $\gamma \cap \mathbb{R}$, $4 < \kappa < 8$. It is an open problem to determine the Hausdorff dimension of the set of double-points of SLE$_\kappa$, $4 < \kappa < 8$.

There is still a lot to be done to further strengthen the links between SLE and CFT. One broad area involves rigorously proving predictions about critical exponents and other “observables” for various 2d models.

Convergence of Multiple LERW to Multiple SLE_2

- SLE can often be used to calculate “observables” such as crossing probabilities (such as Cardy’s formula) and (non-)intersection probabilities.
- These calculations are often a crucial step in proving convergence of discrete models to SLE.
- Although the following result is general, it should help in the particular case of proving multiple LERW converges to multiple SLE_2.

Convergence of Multiple LERW to Multiple SLE$_2$



Theorem. (K. 2009) Suppose that $0 < x < y < \infty$ are real numbers and let $\beta : [0, t_\beta] \to \overline{\mathbb{H}}$ be a Brownian excursion from x to y in $\mathbb{H}$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$, $0 < \kappa \le 4$, from 0 to ∞ in $\mathbb{H}$, then

$$\mathbf{P}\{\,\gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset\,\} = \frac{\Gamma(2a)\Gamma(4a+1)}{\Gamma(2a+2)\Gamma(4a-1)}\,(x/y)\,F(2, 1-2a, 2a+2; x/y)$$

where $F = {}_2F_1$ is the hypergeometric function and $a = 2/\kappa$.