Partial results on the convergence of loop-erased random walk to SLE(2) in the natural parametrization

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37th Conference on Stochastic Processes and their Applications
Universidad de Buenos Aires

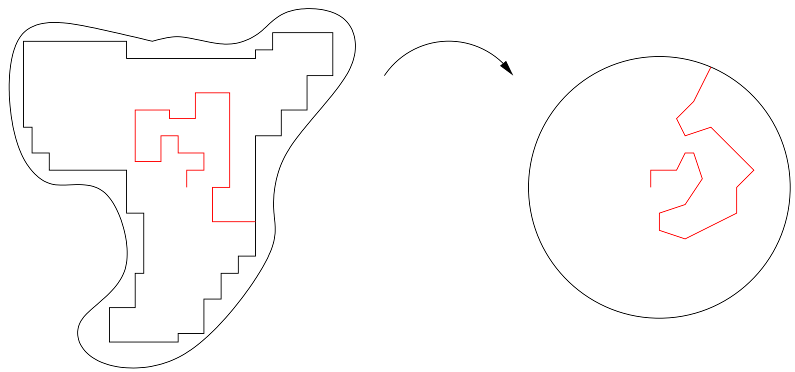
July 28, 2014

Based on joint work with Tom Alberts and Robert Masson
J. Stat. Phys. 153:119–141, 2013.

Introduction

The plan is to discuss a strategy for showing convergence of loop-erased random walk on the two-dimensional square lattice to SLE(2), in the supremum norm topology that takes the time parametrization of the curves into account.

From LERW to SLE



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$; that is, D_n is the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting ∂D.
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\hat{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parameterize by capacity.

Loop-Erased Random Walk Converges to SLE(2)

Consider the following metric on the space of curves in $\mathbb{C}$:

$$\rho(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{0 \le t \le 1} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)|$$

where the infimum is over all choices of parametrizations $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $[0,1]$ of γ_1 and γ_2.

Let μ_n denote the law of γ_n, time-reversed LERW from 0 to ∂D_n, and let μ denote the law of the image in D of radial SLE(2).

Theorem. (Lawler-Schramm-Werner)
The measures μ_n converge weakly to μ as $n \to \infty$ with respect to the metric ρ on the space of curves.

Important. This theorem tells us that the LERW and SLE(2) traces are close. It does not tell us that they are close in space at roughly the same time.

Our Goal

Suppose that X is a LERW on $\mathbb{Z}^2$ started at the origin and let τ_n be the first time the curve hits the circle of radius n. We would like

(i) to show that there is a speed function $t \mapsto \sigma_n(t)$ so that

$$t \mapsto \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

converges in law under the strong topology, and

(ii) to identify the limiting curve as SLE(2) in the natural time parametrization that was recently introduced by Lawler-Sheffield and Lawler-Zhou.

Outline

- To discuss a strategy for (i) proving that the limit exists.
- To discuss a strategy for (ii) identifying the limit.
- We'll see how to choose the speed function $\sigma_n(t)$ to execute both strategies

Strategy for (i) Proving that the Limit Exists

Prove tightness!

There are a number of techniques for proving tightness of a stochastic process.

But. . . most of them were designed for Markov processes.

So we'll move to a different setting using an occupation measure.

An Occupation Measure

If γ is a curve, then its occupation measure ν_γ identifies the amount of time γ spends in each Borel subset of $\mathbb{C}$.

Formally,

$$\nu_\gamma(A) := \int_0^{t_\gamma} 1\{\gamma(s) \in A\} \, \mathrm{d}s$$

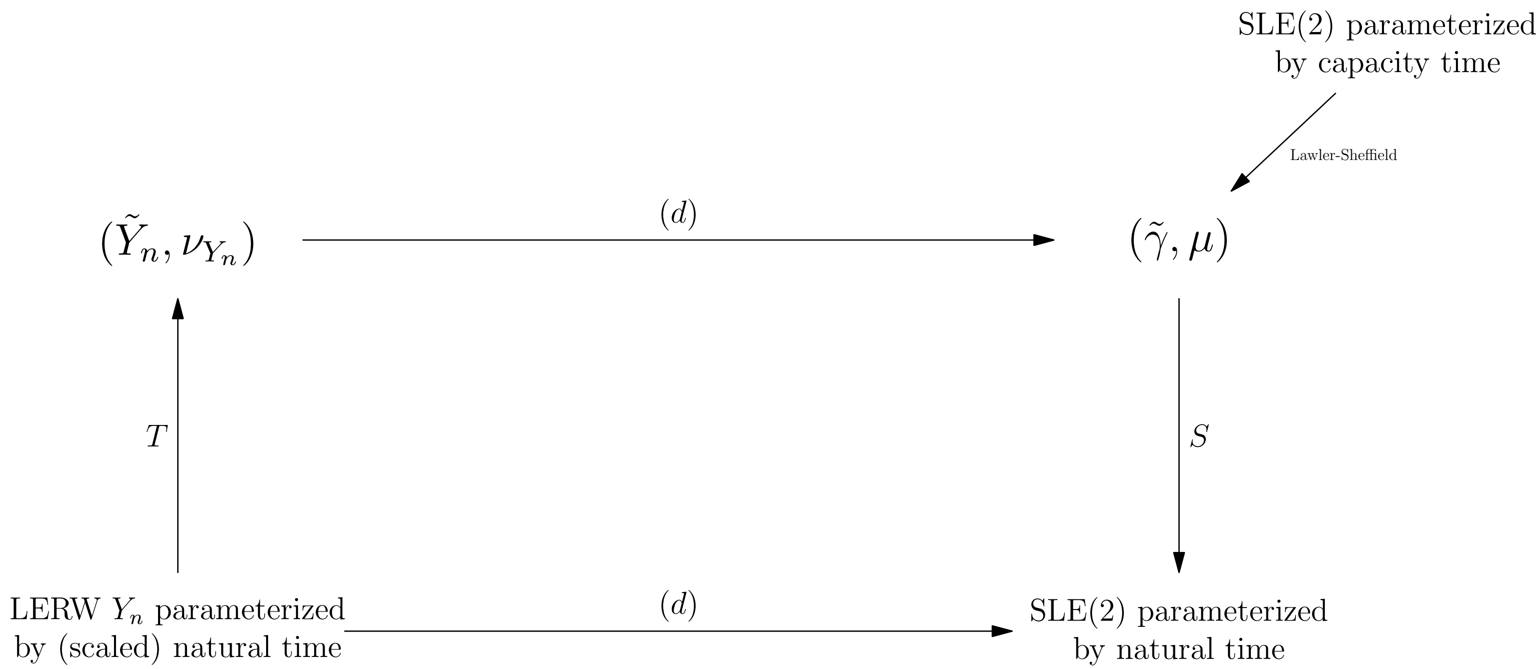
where A is a Borel subset of $\mathbb{C}$.

Note. Implicit in the statement that γ is a curve is its time parametrization.

- ν_γ is supported on γ
- The total mass of ν_γ is t_γ

Key observation.

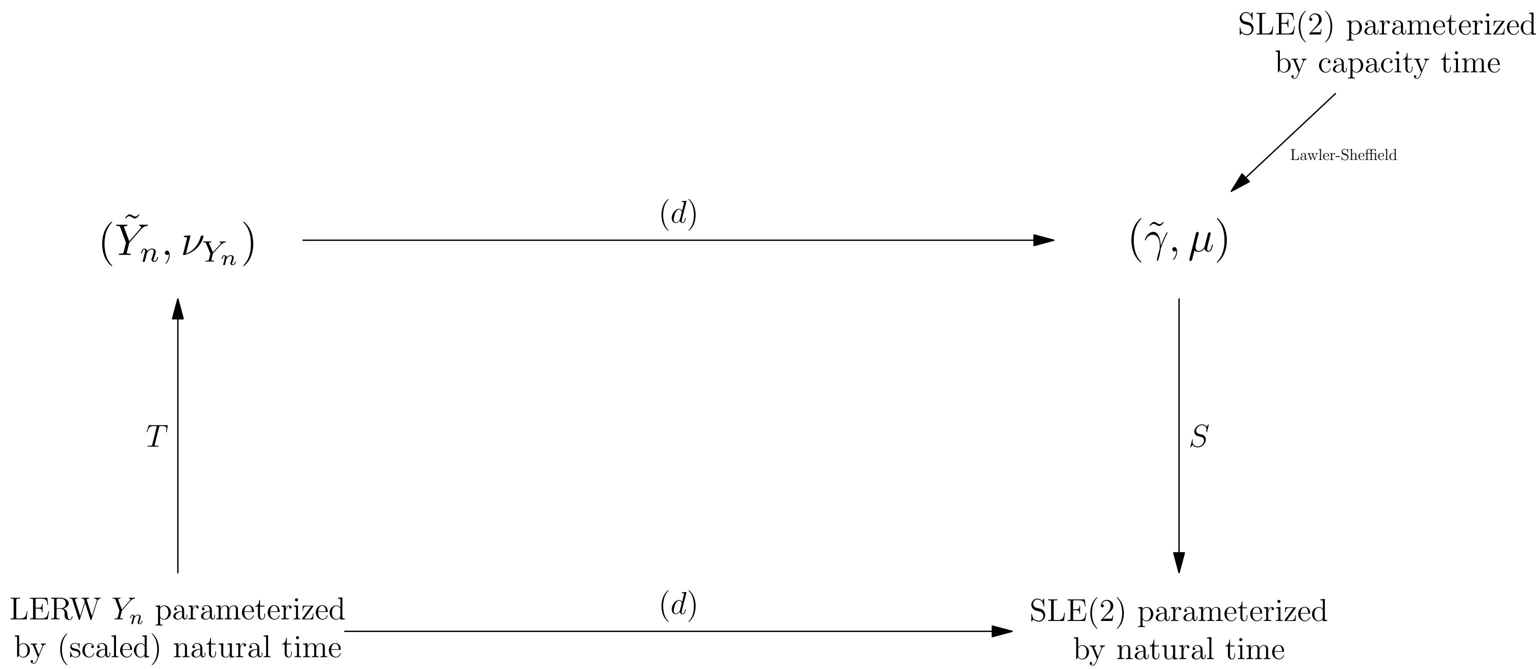
occupation measure + curve modulo reparametrization $\Rightarrow$ original curve



$$Y_n = \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

The topology on the top is the product topology $\tilde{\Omega} \times \mathcal{M}$: equivalence classes of curves induced by ρ along with weak convergence on the space of positive Borel measures on $\mathbb{C}$.

Convergence on top implies convergence on bottom if T and S are continuous. T is actually Lipshitz, but S is not continuous (or even well-defined) but it is at all the limit points we will encounter.



$$Y_n = \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Strategy: prove tightness for $(\tilde{Y}_n, \nu_{Y_n})$, then prove uniqueness of subsequential limits.

Advantage: the $\tilde{Y}_n \to \tilde{\gamma}$ part has already been done! (LSW)

For tightness of ν_{Y_n}, it is sufficient to prove that the lifetimes of Y_n are tight.

Consequence: Loop-Erased Random Walk Converges to SLE(2)

Let γ be a radial SLE(2) started uniformly on $\partial\mathbb{D}$.

Suppose that $X(t)$, $0 \le t \le M_n$, denotes the time reversal of a loop-erased random walk on $\mathbb{Z}^2$ started at the origin and stopped at M_n, the time the loop-erased random walk reaches the circle of radius n.

If $z \in \mathbb{D}$, $\epsilon > 0$, and

$$Y_n(t) = \frac{1}{n} X(\sigma_n(t))$$

where $\sigma_n(t)$ is a speed function, then

$$\lim_{n\to\infty} \mathbb{P}\left(\tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right) = \mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right).$$

Strategy for (ii) Identifying the Limit

If γ is SLE in the natural time parametrization, then $(\tilde{\gamma}, \nu_\gamma)$ has certain natural properties, namely it satisfies conformal convariance and the domain Markov property.

In fact, $(\tilde{\gamma}, \nu_\gamma)$ is the unique pair having both properties.

Given tightness, the strategy is to show that all subsequential limits have these properties.

It mimics the original LSW proof.

Properties of $(\tilde{\gamma}, \nu_\gamma)$

(We conjecture that) there is a unique probability measure on $\tilde{\Omega} \times \mathcal{M}$ such that for a pair $(\tilde{\gamma}, \mu)$,

- $\tilde{\gamma}$ is SLE(2) in the unit disk $\mathbb{D}$,
- μ is measurable with respect to $\tilde{\gamma}$,
- if $\gamma \in \tilde{\gamma}$, then $\mu(\cdot \cap \gamma[0,t])$ is measurable wrt $\widetilde{\gamma}[0,t]$,
- $\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$ where G is the Green's function for SLE defined by

$$G(z) = \lim_{\epsilon \to 0+} \epsilon^{-3/4} \mathbf{P}\left\{\gamma \cap B(z;\epsilon) \neq \emptyset\right\},$$

  and
- the domain Markov property holds for μ.

Uniqueness is easy, but existence is hard.

Strategy for (ii) Identifying the Limit

Show that all subsequential limits $(\tilde{\gamma}, \mu)$ of $(\tilde{Y}_n, \nu_{Y_n})$ have the properties that

- $\tilde{\gamma}$ is SLE(2) in the unit disk $\mathbb{D}$,
- μ is measurable with respect to $\tilde{\gamma}$,
- if $\gamma \in \tilde{\gamma}$, then $\mu(\cdot \cap \gamma[0,t])$ is measurable wrt $\widetilde{\gamma}[0,t]$,
- $\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$ where G is the Green's function for SLE, and
- the domain Markov property holds for μ.

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Most desirable choice is $\sigma_n(t) = n^{5/4} t$

Based on the long-standing conjecture that M_n "grows like" $n^{5/4}$ where M_n is the number of steps in the LERW (i.e., $M_n = \tau_n$)

Very, very difficult to prove! This would imply that

$$\frac{M_n}{n^{5/4}}$$

has a limiting distribution as $n \to \infty$.

Strongest known result is still that

$$\lim_{n\to\infty} \frac{\log M_n}{\log n} = \frac{5}{4}.$$

(Originally proved by Kenyon, later by Masson.)

But we don't even know how to get tightness!

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Second choice is $\sigma_n(t) = \mathbf{E}[M_n]t$

This implies that the total lifetime of Y_n is $M_n/\mathbf{E}[M_n]$

Barlow and Masson give tightness bounds for this. In fact, they also give exponential tail bounds

$$\mathbf{P}\left\{\alpha^{-1} \leq \frac{M_n}{\mathbf{E}[M_n]} \leq \alpha\right\} \geq 1 - Ce^{-c\alpha^{1/2}}.$$

"Historical" remark: This result is what really motivated the present work.

Another advantage: If this works, then showing convergence for the first choice of speed function reduces to showing that

$$\mathbf{E}[M_n] \sim cn^{5/4}.$$

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

So let's use our second choice:

$$\sigma_n(t) = \mathbf{E}[M_n]t.$$

There are five properties that all subsequential limits need to satisfy. The measurability properties seem okay.

But, we still need to show that all subsequential limits satisfy conformal covariance and the domain Markov property.

Let's focus on trying to prove that

$$\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$$

for all subsequential limits μ of ν_{Y_n}.

How should the speed function by chosen?

In the special case of a ball $B(z;\epsilon) \subset \mathbb{D}$, we believe that the simple geometry can be helpful. Write

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon))\right] = \mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid \tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right] \mathbb{P}\left(\tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right).$$

The second term on the right above converges to $\mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right)$.

For the first term, we roughly expect that the loop-erased walk goes through $B(z;\epsilon)$ as if it were a loop-erased walk in that domain, i.e., it should not be influenced too much by its future or past. This leads to the conjecture that

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] = \frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} + o(1).$$

How should the speed function by chosen?

To complete the convergence to the Green's function via this strategy, we also expect that

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} = \epsilon^{5/4}[1 + o(1)]$$

as $\epsilon \to 0$.

Combining with the previous estimates, this will show that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon))\right] &= \epsilon^{5/4}\mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right)[1 + o(1)] \\ &= \epsilon^2 G(z)[1 + o(1)].\end{aligned}$$

How should the speed function by chosen?

Theorem. If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] \leq C \log(1/\epsilon)\epsilon^{5/4}.$$

Conjecture. If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] = \frac{\mathbf{E}[M_{\epsilon n}]}{\mathbf{E}[M_n]} + o(1)$$

as $n \to \infty$.

To prove the conjecture, need a strong separation lemma. This is currently out of reach.

Says that the curve up until it hits the ball of radius ϵ does not too strongly affect how the curve behaves inside the ball of radius ϵ.

How should the speed function by chosen?

Write $\sigma_n(t) = c_n t$.

It is sufficient to prove that

$$\sum_{e \in A_n} \left[\frac{2n^2}{c_n} \mathbb{P}\left(z_e \in \tilde{Y}_n \right) - G(z_e) \right] = o(n^2)$$

where $A_n = A \cap n^{-1}\mathbb{Z}^2$ so that the sum is over all (undirected) edges e of A_n, and z_e is the midpoint of the edge e.

Carrying out this estimate appears to be genuinely difficult. It is a hard problem to describe asymptotics for the probability that loop-erased walk passes through a particular edge, and even harder to show that the limit is the SLE Green's function.

Recently, C. Beneš, G. Lawler, and F. Johansson Viklund proved that the chordal loop-erased random walk Green's function converges to the chordal SLE Green's function. However, it is not clear if their techniques can be modified to work in the radial case.