# A STUDY ON THE HÁJEK-RÉNYI INEQUALITY AND ITS APPLICATIONS

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# Abstract

Inequalities are at the heart of mathematical and statistical theory. No inequality is completely perfect, but the Hájek-Rényi inequality, which is the main subject of this thesis, is arguably the closest to absolute perfection of all the inequalities within all theories of probability. It has many applications in proving limit theorems, and examples of these are presented in this thesis. The strong law of large numbers for sequences of random variables and the strong growth rate for sums of random variables were obtained through utilizing the Hájek-Rényi inequality. This thesis will further extend and improve the proof of the strong law of large numbers. Additionally, the approach, utilizing the Hájek-Rényi inequality to prove limit theorems, is also applied to the weak law of large numbers for tail series.

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# Contents

# Chapter 1

# Introduction

The strong law of large numbers allows for the transformation of a sequence made up of cumulative sums of random variables into a nonrandom sequence. This is completed through normalizing the initial sequence using a sequence of nonrandom numbers before approaching the limit. Proving the strong law of large numbers can be done by determining the desired result for a particular subsequence, before taking the whole sequence and reducing the problem, so that the subsequence result is applicable. The determination of a maximal inequality for the whole sequence of cumulative sums is necessary. Previous research in probability theory has found numerous maximal inequalities for various classes of random variables. Therefore, individual determination may not be necessary. An alternative method to prove the strong law of large numbers is more difficult, but can be done by applying a maximal inequality for normalized sums. This inequality is referred to as the Hájek-Rényi inequality, in honour of a paper written by Hájek and Rényi (1955) describing independent summands. This thesis will illustrate that the Hájek-Rényi inequality is a result of choosing the correct maximal inequality for a sequence of cumulative sums.

## 1.1 Literature review

Discussion will begin in the literature pertaining to the Hájek-Rényi inequality. Most of the research will focus on the strong law of large numbers, which applies to the main topic of this thesis. We start with Hájek-Rényi (1955). The authors proved the following important inequality: if $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$, and $\{b_n, n \geq 1\}$ is a positive nondecreasing real sequence, for any $\epsilon > 0$, any positive integer $m < n$,

$$P\left(\max_{m \leq k \leq n} \left| \frac{\sum_{j=1}^{k} X_j}{b_n} \right| \geq \epsilon \right) \leq \epsilon^{-2} \left( \sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2} \right).$$

The following result is proved by Chow (1960). Let $\{y_k, k \geq 1\}$ be a semimartingale, let $c_1 \geq c_2 \geq \cdots$ be positive constants, and let $\varepsilon > 0$, then

$$\varepsilon P\{\max_{m \geq k \geq 1} c_k y_k \geq \varepsilon\} \leq \sum_{1}^{m-1} (c_k - c_{k+1}) E\{{y_k}^+\} + c_m E\{{y_m}^+\}.$$

where $y^+ = \max[y, 0]$. This inequality reduces to one proved by Hájek-Rényi (1955). Three theorems are proved by Bickel (1969). Let $\{X_k\}$ be a sequence of independent random variables, $\{S_k\}$ the sequence of their partial sums, and $\{C_k\}$ a decreasing sequence of positive real numbers. Let $g$ be a positive convex function.

Theorem 1: If the $X_k$ have a symmetric distribution, then

$$P[\max_{1\leq k\leq n} C_k g(S_k) \geq \varepsilon] \leq 2P[\{C_n g(S_n) + \sum_{j=1}^{n}(c_j - c_{j+1})g(S_j)\} \geq \varepsilon].$$

Theorem 2: Let $S(t)$ be a stochastic process with symmetric independent increments on an interval $[a, b]$, and let the function $-c(t) + c(a)$ generate a positive measure on $[a, b]$, then

$$P[\sup_{a<t<b} c(t)g(S(t)) \geq \varepsilon] \leq 2P[\{\int_{a^+}^{b^-} g(S(t))\, d[-c(t)] + c(b) \limsup_{t\to b} g(S(t))\} \geq \varepsilon].$$

Theorem 3: If, in addition to the conditions imposed in Theorem 2, the $X_k$ has zero expectation and $g$ is even and satisfies $g(x + y) \leq K\{g(x) + g(y)\}$, for all real $x, y$ and some constant $K$, then

$$E(\sup_n c_n g(S_n)) \leq 4K[\sum_{1}^{\infty}(c_n - c_{n+1})E(g(S_n))] + \limsup c_n E(g(S_n)).$$

An inequality, similar to the Hájek-Rényi inequality, is derived by Sen (1972). The inequality is based on a decomposition of the $U$-statistic and a semi-martingale extension of the Hájek-Rényi inequality by Chow (1960). Moreover, the Kolmogorov type inequality for generalized $U$-statistics is presented. An extension of the well-known Hájek-Rényi inequality for the sum of a sequence of random variables, which does not involve any moment conditions or the assumption of independence, is obtained by Szynal (1973). This extension allows him to establish the almost sure convergence of certain sequences of random variables which other known inequalities are unable to accomplish. Chandra and Ghosal (1996) introduce two classes of sequences of random variables: the class of asymptotically quadrant sub-independent sequences (AQSI) and the class of asymptotically almost negatively associated sequences (AANA). By definition, $\{X_n, n \geq 1\} \in$ AQSI if there exists a nonnegative sequence $\{q(m), m \geq 1\}$ such that, for all distinct $i$ and $j$:

$$P[X_i > s, X_j > t] - P[X_i > s]P[X_j > t] \leq q(|i-j|)\alpha_{i,j}(s,t), \quad s,t \geq 0;$$

and

$$P[X_i < s, X_j < t] - P[X_i < s]P[X_j < t] \leq q(|i-j|)\beta_{i,j}(s,t), \quad s,t \leq 0.$$

where $q(m) \to 0$ as $n \to \infty$ and $\alpha_{i,j}(s,t) \geq 0, \beta_{i,j}(s,t) \geq 0$. However, $\{X_n, n \geq 1\} \in$ AANA if there is a nonnegative sequence $q(m) \to 0$ such that

$$\mathrm{cov}(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \leq q(m)(\mathrm{var}(f(X_m))\mathrm{var}(g(X_{m+1}, \cdots, X_{m+k})))^{1/2}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions $f$ and $g$, whenever the right-hand side is finite.

In Gan (1997), a $p$-smoothable Banach space is characterized in terms of the Hájek-Rényi inequality for Banach space valued martingales. As applications of this inequality, the strong law of large numbers and integrability of the supremum of Banach space valued martingales are also given. In Liu, Gan, and Chen (1999), the Hájek-Rényi inequality is obtained and the Marcinkiewicz-Zygmund strong law of large numbers for negatively associated random variables is discussed. In particular, for independent and identically distributed random variables, the classical Marcinkiewicz strong law of large numbers is generalized to the case of negatively associated random variables. The material from this paper is discussed in detail in Chapter 2 of this thesis.

In Fazekas and Klesov (2002), the authors obtain a maximal inequality for the partial sums, $S_n$, of random variables without any assumptions of independence or distribution on the random variables. Using the maximal inequality, they prove the following strong law of large numbers: Let $\{b_n\}$ be a nondecreasing unbounded sequence of positive numbers and $\{\alpha_n\}$ be a sequence of non-negative numbers. Assume that

$$E\left(\max_{1\leq k\leq n}|S_k|\right)^r \leq \sum_{k=1}^{n}\alpha_k \quad \text{and} \quad \sum_{k=1}^{\infty} a_k/b_k^r < \infty,$$

for all $n \geq 1$ and a fixed number $r > 0$. Then $\lim_{n\to\infty} S_n/b_n = 0$ a.s. In Cai (2000), the Hájek-Rényi inequality for $\rho^*$-mixing sequences of random variables is proved. The material from this paper is discussed in detail in Chapter 3. In Yang and Su (2000), a general method based on the Hájek-Rényi inequality for establishing the strong law of large numbers is provided. The material from this paper is discussed in detail in Chapter 6.

Rosalsky and Volodin (2001) study the rate of convergence of the tail of a convergent sequence of random variables. Let $\{V_j\}_{j=1}^{\infty}$ be a sequence of random variables with values in a separable space such that $\sum_{j=1}^{\infty} V_j$ exists a.s.. The authors investigate under which additional conditions $b_n^{-p}(\lim_{m\to\infty} E(\max_{n\leq k\leq m} \| \sum_{j=n}^{k} V_j\|^p)) = O(1)$ implies that $b_n^{-p} \sup_{k\geq n} \| \sum_{j=k}^{\infty} V_j\|^p$ is bounded in probability, where $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive numbers and $p > 0$. The authors generalize the above situation to the case where, instead of $f(x) = |x|^p$, the nondecreasing function $f\colon [0,\infty) \to [0,\infty)$, such that $f(0) = 0$ and $f(x) > 0$ for $x > 0$, is used.

In Rao (2002), the following result is proved. Let $X_1, X_2, \ldots, X_n$ be positively associated random variables. The author deduces an upper bound for

$$P(\max_{1\leq k\leq n} |(1/b_n)\sum_{i=1}^{k}(X_i - EX_i)| \geq \epsilon),$$

where $0 < b_1 \leq b_2 \leq b_3 \leq \ldots$ and $\epsilon > 0$.

In Kim, Ko, and Han (2005), the following result is provided. Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically quadrant subindependent (AQSI) random variables. Under some restrictions on $\alpha_{ij}$, $\beta_{ij}$ and $q$, it is shown that there exists $c > 0$ and a nondecreasing sequence, $\{b_n, n \geq 1\}$, of positive numbers such that for any $\varepsilon > 0$,

$$P\left\{\max_{1\leq k\leq n}\left|\frac{\sum_{i=1}^{k}(X_i - EX_i)}{b_k}\right| \geq \varepsilon \log n\right\} \leq c\left(\frac{\log n\,(\log 3) + 2}{\varepsilon \log n}\right)^2 \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2}.$$

Also, it is shown that $(b_n \log n)^{-1} S_n \to 0$ as $n \to \infty$ a.s. and for any $0 < r < 2$

$$E\sup_n\left(\frac{|S_n|}{b_n \log n}\right)^r < \infty,$$

where $S_n = \sum_{i=1}^{n}(X_i - EX_i)$. This paper is discussed in detail in Chapter 4.

In Ko, Kim, and Lin (2005), the Hájek-Rényi type inequality and the strong law of large numbers are proved for weighted sums of asymptotically almost negatively associated sequences of random variables. This paper is discussed in detail in Chapter 5. In Qiu and Gan (2005), the Hájek-Rényi inequality for NA (negatively associated) and arbitrary random variables are established, and the strong law of large numbers for NA sequences is obtained by using the inequality. In Hu, Hu, and Zhang (2005), the authors show that the Hájek-Rényi type inequality holds for any random variables

$X_1, X_2, \ldots$, if

$$E(\max_{m \le i \le n} |\sum_{j=m}^{i}(X_j - EX_j)/b_j|^2) \le C_1 E(\sum_{j=m}^{n}(X_j - EX_j)/b_j)^2,$$

holds for any $1 \le m \le n$ and a positive constant $C_1$, where $\{b_n\}$ is a nondecreasing positive real-valued sequence.

## 1.2 Organization of the thesis

The thesis is organized as follows. In Chapter 2, the Hájek-Rényi inequality for negatively associated random variables is introduced. The material is based on the paper by Liu, Gan, and Chen (1997). In Chapter 3, the Hájek-Rényi inequality for $\rho^*$-mixing sequences of random variable is introduced. The material is based on the paper by Cai (2000). In Chapter 4, the Hájek-Rényi inequality for asymptotically quadrant subindependent random variables is introduced. The material of this chapter is based on the paper by Kim, Ko, and Han (2005). In Chapter 5, the Hájek-Rényi inequality for asymptotically almost negatively associated random variables is introduced. This chapter is based on the paper by Ko, Kim, and Lin (2005). In Chapter 6, the Hájek-Rényi inequality is utilized, in order to prove the strong law of large numbers for general summands. The material is based on the preprint by Yang, and Su (2000). Chapter 7 contains new contributions to the study of the Hájek-Rényi inequality and the strong law of large numbers.

## 1.3 Contributions of the thesis

Using the Hájek-Rényi type maximal inequality, Fazekas and Klesov (2000) obtained the strong law of large numbers for sequences of random variables. Under the same conditions, as found in Fazekas and Klesov (2000), Hu and Hu (2006) obtained the strong growth rate for sums of random variables which improves the result of Fazekas and Klesov (2000). In the last chapter, we find a new method for obtaining the strong growth rate for sums of random variables through using the approach of Fazekas and Klesov (2000). It allows us to generalize and sharpen the method of Hu and Hu (2006). Our method can be applied to almost all cases of the dependence structure considered in Hu and Hu (2006), and we can obtain better results. Additionally, the approach of using the Hájek-Rényi type maximal inequality to prove limit theorems, is also applied to the weak law of large numbers for tail series.

# Chapter 2

# The Hájek-Rényi Inequality for Negatively Associated (NA) Random Variables

The results of this chapter are based on the paper by Liu, Gan, and Chen (1997).

## 2.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$.

**Definition 2.1.1** *A finite family of random variables* $\{X_1, \ldots, X_n\}$ *is said to be negatively associated (abbreviated to NA) if* $\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$ *for any disjoint subsets* $A, B \subset \{1, \ldots, n\}$ *and any real coordinatewise nondecreasing functions* $f$ *on* $\mathbb{R}^A$ *and* $g$ *on* $\mathbb{R}^B$.

An infinite family of random variables is said to be NA if every finite subfamily is NA.

## 2.2 The Hájek-Rényi inequality for NA random variables

**Theorem 2.2.1** *Let* $\{X_n, n \geq 1\}$ *be NA random variables with* $EX_n^2 < \infty, n \geq 1$ *and* $\{b_n, n \geq 1\}$ *be a positive nondecreasing real number sequence. For* $\varepsilon > 0$*, the following inequality is true*

$$P\left(\max_{1\leq k\leq n}\left|\frac{1}{b_n}\sum_{i=1}^{k}(X_i - EX_i)\right| \geq \varepsilon\right) \leq 32\varepsilon^{-2}\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2},$$

*where* $\sigma_j^2 = \mathrm{Var}(X_j)$*,* $j \geq 1$*, the variance of the random variable* $X_j$*.*

**Proof.** We suppose that $EX_j = 0$, $j \geq 1$, without loss of generality and let $b_0 = 0$, we have

$$S_k = \sum_{j=1}^{k} b_j\frac{X_j}{b_j} = \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i - b_{i-1})\frac{X_j}{b_j}\right) = \sum_{i=1}^{k}(b_i - b_{i-1})\sum_{i\leq j\leq k}\frac{X_j}{b_j}.$$

Note that $\frac{1}{b_k}\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$, then

$$\{|S_k/b_k| \geq \varepsilon\} \subset \{\max_{1\leq i\leq k}|\sum_{i\leq j\leq k}\frac{X_j}{b_j}| \geq \varepsilon\}.$$

Therefore,

$$\{\left|\max_{1\le k\le n}\frac{S_k}{b_k}\right|\ge\varepsilon\}\subset\{\max_{1\le k\le n}\max_{1\le i\le k}\left|\sum_{i\le j\le k}\frac{X_j}{b_j}\right|\ge\varepsilon\}=\{\max_{1\le i\le k\le n}\left|\sum_{j\le k}\frac{X_j}{b_j}-\sum_{j<i}\frac{X_j}{b_j}\right|\ge\varepsilon\}$$
$$\subset\{\max_{1\le i\le k}\left|\sum_{j=1}^{i}\frac{X_j}{b_j}\right|\ge\frac{\varepsilon}{2}\}.$$

Using the Kolmogorov-type inequality of NA random variables (see Matula, 1992), we have

$$P(\max_{k\le n}|\frac{S_k}{b_k}|\ge\varepsilon)\le 32\varepsilon^{-2}\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}.$$

The theorem is proved.

## 2.3 The strong law of large numbers for NA random variables

**Theorem 2.3.1** *Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers and let $\{X_n, n\ge 1\}$ be NA random variables with $\sum_{n=1}^{\infty}\frac{\sigma_n^2}{b_n^2}<\infty$ where $\sigma_n^2 = var X_n, n\ge 1$, for any $0<r<2$,*

1. *$E\sup_n(|S_n|/b_n)^r<\infty$.*

2. *Assume that $0<b_n\uparrow\infty$, then $S_n/b_n\to 0$ a.s.$(n\to\infty)$, where*

$$S_n=\sum_{i=1}^{n}(X_i-EX_i),\ \ n\ge 1$$

**Proof.** For the first part, note that

$$E\sup_n\left(\frac{|S_n|}{b_n}\right)^r < \infty \Leftrightarrow \int_1^\infty P\left(\sup_n\left(\frac{|S_n|}{b_n}\right) > t^{1/r}\right)dt < \infty.$$

Using the Hájek-Rényi inequality (by Theorem 2.2.1), we have

$$\int_1^\infty P\left(\sup_n\left(\frac{|S_n|}{b_n}\right) > t^{1/r}\right)dt \le 32\int_1^\infty t^{-2/r}\sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2}dt = 32\sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2}\int_1^\infty t^{-2/r}dt < \infty.$$

As for the second part, using the Hájek-Rényi inequality (by Theorem 2.2.1), we have

$$P\left(\max_{m\le k\le n}\left|\frac{1}{b_n}\sum_{i=1}^k (X_i - EX_i)\right| \ge \varepsilon\right) \le 128\varepsilon^{-2}\left(\sum_{j=m+1}^n \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^m \frac{\sigma_j^2}{b_m^2}\right).$$

But

$$\begin{aligned} P\left(\sup_n\left|\frac{1}{b_n}\sum_{i=1}^n (X_i - EX_i)\right| \ge \varepsilon\right) &= \lim_{n\to\infty} P\left(\max_{m\le j\le n}\left|\frac{1}{b_n}\sum_{i=1}^j (X_i - EX_i)\right| \ge \varepsilon\right) \\ &\le 128\varepsilon^{-2}\left(\sum_{j=m+1}^\infty \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^m \frac{\sigma_j^2}{b_m^2}\right). \end{aligned}$$

Using the Kronecker lemma, we have

$$\lim_{n\to\infty} P\left(\sup_n \left|\ \frac{1}{b_n}\sum_{i=1}^n (X_i - EX_i)\ \right| \ge \varepsilon\right) = 0,$$

and the theorem is proved.

# Chapter 3

# The Hájek-Rényi Inequality for $\rho^*$-mixing Sequences of Random Variables

The results of this chapter are based on the paper by Cai (2000).

## 3.1 Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $\{X_i, i \geq 1\}$ be a sequence of random variables defined on it. We denote that a random variable $X \in L_2(\mathcal{F})$, if $X$ is $\mathcal{F}$ measurable and $EX^2 < \infty$.

Let $S, T \subset \mathcal{N}$ be nonempty subsets. Define $\mathcal{F}_{\mathcal{S}} = \sigma(X_k, k \in S)$, $\sigma$-algebra generated by these random variables. The *maximal correlation coefficient* is defined as $\rho_n^* = \sup corr(f, g)$, where the supremum is taken over all $(S, T)$ with $dist(S, T) \geq n$ and all $f \in L_2(\mathcal{F}_S), g \in L_2(\mathcal{F}_T)$, where $dist(S, T) = inf_{x \in S, y \in T}|x - y|$.

A sequence of random variables $\{X_n, n \geq 1\}$ is called $\rho^*$-*mixing* if $\lim_{n\to\infty} \rho_n^* < 1$.

## 3.2 The Hájek-Rényi inequality for $\rho^*$-mixing sequence

Before we formulate the main result of this section, we need a few lemmas.

**Lemma 3.2.1** *Let $\{X_i, i \geq 1\}$ be a $\rho^*$-mixing sequence of random variables with $EX_i = 0, E|X_i|^p < \infty$ for some $p \geq 2$, and all $i \geq 1$. Then there exist $C = C(p)$, such that*

$$E \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_i\right|^p \leq C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{3.1}$$

**Proof**. See Utev and Peligrad (2003).

**Lemma 3.2.2** *Let $\{\beta_i, i \geq 1\}$ be a nondecreasing sequence of positive numbers and $\alpha_1, \cdots, \alpha_n$ be nonnegative numbers. Let $p$ be a fixed positive number. Assume that for each $m$ with $1 \leq m \leq n$,*

$$E \max_{1\leq k\leq m} \left|\sum_{i=1}^{k} X_i\right|^p \leq \sum_{i=1}^{m} \alpha_i.$$

*Then*

$$E \max_{1\leq k\leq n} \left|\frac{\sum_{i=1}^{k} X_i}{\beta_k}\right|^p \leq 4\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i^p}.$$

**Proof**. See Fazekas and Klesov (2000).

Now we can formulate and prove the main result of this section, that is the Hájek-Rényi inequality for $\rho^*$-mixing sequence.

**Theorem 3.2.1** *Let $\{X_i, i \geq 1\}$be a $\rho^*$-mixing sequence of random variables with $EX_i = 0, EX_i^2 < \infty$. Let $\{\beta_i, i \geq 1\}$ be a nondecreasing sequence of positive numbers. Then*

$$E \max_{1 \leq k \leq n} \left| \frac{\sum_{i=1}^k X_i}{\beta_k} \right|^2 \leq C \sum_{i=1}^n \frac{EX_i^2}{\beta_i^2}.$$

**Proof**. The proof is obvious, if we use two lemmas presented above with $p = 2$.

## 3.3 Applications to convergence of random series and supremum of partial sums

The first theorem shows how the Hájek-Rényi inequality, presented in the previous section, can be applied to the study of strong convergence of series of random variables.

**Theorem 3.3.1** *Let $\{X_i, i \geq 1\}$be a $\rho^*$-mixing sequence of random variables with mean zero and $\sum_{i=1}^\infty EX_i^2 < \infty$. Then, the series $\sum_{i=1}^\infty X_i$ converges almost surely.*

**Proof**. By (3.1), for all $m \geq n \geq 1$, we have

$$E|S_m - S_n|^2 \leq C \sum_{i=n+1}^m EX_i^2 \to 0,$$

when $n \to \infty$. Hence, $\{S_n, n \geq 1\}$ is a Cauchy sequence in $L^2$. Therefore, there exists a random variable S with $E|S_n - S|^2 \to 0$, when $n \to \infty$. Using Chebyschev's inequality and Theorem 3.2.1 of previous section, we have for all $\varepsilon > 0$,

$$P(|S_{2^k} - S| > \varepsilon) \leq \varepsilon^{-2} E|S_{2^k} - S|^2 \leq \varepsilon^{-2} \lim_{n \geq 2^k} \sup E|S_{2^k} - S_n|^2 \leq C \sum_{i=2^k+1}^{\infty} EX_i^2 \leq \frac{C}{k^2}.$$

Using the Hájek-Rényi inequality for $\rho^*$-mixing sequence, we have

$$P\left(\max_{2^{k-1} < n \leq 2^k} |S_n - S_{2^{k-1}}| > \varepsilon\right) \leq \varepsilon^{-2} E\left(\max_{2^{k-1} < n \leq 2^k} |S_n - S_{2^{k-1}}|\right)^2 \leq \sum_{i=2^{k-1}+1}^{2^k} EX_i^2.$$

By the last two inequalities, when $k \to \infty$, we have that $S_{2^k} \to S$ a.s. and $\max_{2^{k-1} < n \leq 2^k} |S_n - S_{2^{k-1}}| \to 0$ a.s.. Using the method of sub-sequences, we have that $S_n \to S$ a.s., which completes the proof of the theorem.

The next theorem deals with the supremum of partial sums.

**Theorem 3.3.2** *Let $\{X_i, i \geq 1\}$ be a $\rho^*$-mixing sequence of random variables with mean zero and $\sum_{i=1}^{\infty} b_i^{-2} EX_i^2 < \infty$, where $\{b_i, i \geq 1\}$ be a positive non-decreasing sequence of real numbers. Then for any $0 < r < 2$, $E \sup_{n \geq 1} (b_n^{-1} |S_n|)^r < \infty$.*

**Proof**. Note that $E \sup_{n \geq 1} (b_n^{-1} |S_n|)^r < \infty$ if and only if

$$\int_1^{\infty} P(\sup_{n \geq 1} (b_n^{-1} |S_n| > t^{1/r}) dt < \infty.$$

Using the Hájek-Rényi inequality for $\rho^*$-mixing sequence, it follows that

$$
\begin{aligned}
\int_1^\infty P\left(\sup_{n\geq 1}(b_n^{-1}|S_n| > t^{1/r}\right) dt &\leq \int_1^\infty t^{-2/r} E(\sup_{n\geq 1} b_n^{-2} S_n^2) dt \\
&= \int_1^\infty t^{-2/r} \lim_{N\to\infty} \sum_{i=1}^N E \max_{2^{i-1}<k\leq 2^i} b_k^{-2} S_k^2 dt \\
&\leq C \int_1^\infty t^{-2/r} \lim_{N\to\infty} \sum_{i=1}^N \sum_{k=2^{i-1}-1}^{2^i} b_k^{-2} E X_k^2 dt \\
&\leq C \int_1^\infty t^{-2/r} \sum_{k=1}^\infty b_k^{-2} E X_k^2 dt \\
&\leq C \int_1^\infty t^{-2/r} dt \\
&< \infty.
\end{aligned}
$$

# Chapter 4

# The Hájek-Rényi Inequality for Asymptotically Quadrant Sub-Independent (AQSI) Random Variables

The results of this chapter are based on the paper by Kim, Ko, and Han (2005). In this chapter, we derive the Hájek-Rényi inequality for AQSI random variables. We also use this inequality to obtain the strong law of large numbers and some results of the integrability of supremum for a sequence of AQSI random variables.

## 4.1 Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on it. Lehmann (1966) introduced the notion of positive quadrant dependance. We say that a sequence $\{X_n, n \geq 1\}$ of random variables is *negatively dependent* if, for all $s, t \in \mathbf{R}$,

$$P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \geq 0, \tag{4.1}$$

and

$$P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \geq 0. \tag{4.2}$$

Using the magnitude of the left hand side in (4.1) and (4.2) as a measure of dependence, Birkel (1992) introduced the notion of asymptotic quadrant independence. Let $\{q(m), m \geq 1\}$ be a sequence of positive constants such that $q(m) \to 0$ and $\alpha_{ij}(s,t)$, $\beta_{ij}(s,t)$ be nonnegative functions. A sequence $\{X_n, n \geq 1\}$ of random variables is called *asymptotically quadrant independent* (AQI) if for all $i \neq j$ and $s, t \in \mathbf{R}$,

$$|P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\}| \leq q(|i-j|)\alpha_{ij}(s,t), \tag{4.3}$$

$$|P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\}| \leq q(|i-j|)\beta_{ij}(s,t). \tag{4.4}$$

Chandra and Ghosal (1996b) considered a dependence condition which is a useful weakening of this definition of AQI, proposed by Birkel (1992). Let $\{q(m), m \geq 1\}$ be a sequence of positive constants such that $q(m) \to 0$ and $\alpha_{ij}(s,t), \beta_{ij}(s,t)$ be nonnegative functions. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *asymptotically quadrant sub-independent (AQSI)* , if for all $i \neq j$,

$$P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \leq q(|i-j|)\alpha_{ij}(s,t), s, t > 0, \qquad (4.5)$$

$$P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \leq q(|i-j|)\beta_{ij}(s,t), s, t > 0. \qquad (4.6)$$

This AQSI condition is satisfied by asymptotically quadrant independence sequence.

## 4.2 The Hájek-Rényi inequality for AQSI sequences

First we provide a few lemmas.

**Lemma 4.2.1** *Let $X_1, X_2, \cdots, X_n$ be square integrable random variables and let there exist $a_1^2, \cdots, a_n^2$ satisfying*

$$E(X_{m+1} + \cdots + X_{m+p})^2 \leq a_{m+1}^2 + \cdots + a_{m+p}^2, \qquad (4.7)$$

*for all $m, p \geq 1, m + p \leq n$. We have*

$$E\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} X_i\right)^2\right) \leq ((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} a_i^2. \tag{4.8}$$

**Proof**. See the proof of Theorem 10 in Chandra and Ghosal (1996b).

**Lemma 4.2.2** *If $\{X_n, n \geq 1\}$ is a sequence of AQSI and $\{f_n, n \geq 1\}$ is a sequence nondecreasing (nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is also a sequence of AQSI random variables.*

**Proof**. See Chandra and Ghosal (1996b).

Now, we present main result of this section.

**Theorem 4.2.1** *Let $\{X_n, n \geq 1\}$ is a sequence of AQSI random variables such that $EX_n^2 < \infty, n \geq 1, \sum_{m=1}^{\infty} q(m) < \infty$, and for all $i \neq j$*

$$\int_0^\infty \int_0^\infty \alpha_{ij}(s,t)\ ds\ dt \leq D(1 + EX_i^2 + EX_j^2), \tag{4.9}$$

$$\int_0^\infty \int_0^\infty \beta_{ij}(s,t)\ ds\ dt \leq D(1 + EX_i^2 + EX_j^2), \tag{4.10}$$

*where $\alpha_{ij}(s,t) \geq 0$ and $\beta_{ij}(s,t) \geq 0$ and $D$ is a positive constant. Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Then for $\epsilon > 0$, we have*

$$P\left\{\max_{1\leq k\leq n}\left|\frac{\sum_{i=1}^{k}(X_i - EX_i)}{b_k}\right| \geq \epsilon \log n\right\}$$

$$\leq C(\epsilon \log n)^{-2} \left((\log n/\log 3) + 2\right)^2 \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2}. \tag{4.11}$$

**Proof.** Suppose that $X_n^+ = \max\{X_n, 0\}$ and $X_n^- = \max\{-X_n, 0\}$. It is easy to see $\{X_n^+\}$ and $\{X_n^-\}$ form AQSI sequences by Lemma 4.2.2. Using Lemma 2 of Lehmann (1966),

$$Cov(X_i^+, X_j^+) \leq Dq(|i-j|)(1 + EX_i^2 + EX_j^2).$$

Hence,

$$\text{Var}\left(\sum_{i=1}^{n} b_i^{-1} X_i^+\right) \leq C \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2},$$

for all $n \geq 1$. Similarly,

$$\text{Var}\left(\sum_{i=1}^{n} b_i^{-1} X_i^-\right) \leq C \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2},$$

for all $n \geq 1$. Thus,

$$\text{Var}\left(\sum_{i=1}^{n} b_i^{-1} X_i\right) \leq 2\text{Var}\left(\sum_{i=1}^{n} b_i^{-1} X_i^+\right) + 2\text{Var}\left(\sum_{i=1}^{n} b_i^{-1} X_i^-\right)$$

$$\leq C \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2}, \text{ for all } n \geq 1. \tag{4.12}$$

Without loss of generality, we assume $b_0 = 0$ and let $T_k = \sum_{j=1}^{k}(X_j - EX_j)$. We have

$$T_k = \sum_{j=1}^{k} b_j \frac{(X_j - EX_j)}{b_j} = \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i - b_{i-1})\frac{X_j - EX_j}{b_j}\right)$$
$$= \sum_{i=1}^{k}(b_i - b_{i-1}) \sum_{i \le j \le k} \frac{X_j - EX_j}{b_j}.$$

Note that $(1/b_k)\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$. Hence, the event

$$\left\{|\frac{T_k}{b_k}| \ge \epsilon \log n\right\} \subset \left\{\max_{1 \le i \le k}\left|\sum_{i \le j \le k}\frac{X_j - EX_j}{b_j}\right| \ge \epsilon \log n\right\}.$$

Therefore,

$$\left\{\max_{1 \le k \le n}\left|\frac{T_k}{b_k}\right| \ge \epsilon \log n\right\} \subset \left\{\max_{1 \le k \le n}\max_{1 \le i \le k}\left|\sum_{i \le j \le k}\frac{X_j - EX_j}{b_j}\right| \ge \epsilon \log n\right\}$$
$$= \left\{\max_{1 \le i \le k \le n}\left|\sum_{j \le k}\frac{X_j - EX_j}{b_j} - \sum_{j < i}\frac{X_i - EX_j}{b_j}\right| \ge \epsilon \log n\right\}$$
$$\subset \left\{\max_{1 \le i \le n}\left|\sum_{j=1}^{i}\frac{X_j - EX_j}{b_j}\right| \ge \frac{\epsilon}{2}\log n\right\}.$$

By Lemma 4.2.1 and (4.6), we obtain

$$P\left\{\max_{1 \le k \le n}\left|\frac{T_k}{b_k}\right| \ge \epsilon \log n\right\} \le C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 \sum_{i=1}^{n}\frac{1 + EX_i^2}{b_i^2}.$$

The theorem is proved.

From the theorem above, we can obtain the following Hájek-Rényi inequality for AQSI random variables.

**Theorem 4.2.2** *Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $EX_n^2 < \infty$, $n \geq 1$, $\sum_{m=1}^{\infty} q(m) < \infty$ and satisfying (4.9) and (4.10). Then for any $\epsilon > 0$ and for any positive integer $m < n$,*

$$P\left\{\max_{m \leq k \leq n} \left|\frac{\sum_{i=1}^{k}(X_i - EX_i)}{b_k}\right| \geq \epsilon \log n\right\}$$
$$\leq C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 \left(\sum_{j=m+1}^{n} \frac{1 + EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{1 + EX_j^2}{b_m^2}\right).$$

**Proof.** Using the previous theorem, we have

$$P\left\{\max_{m\le k\le n}\left|\frac{\sum_{i=1}^{k}(X_i-EX_i)}{b_k}\right|\ge \epsilon\log n\right\}$$

$$\le P\left\{\left|\frac{\sum_{i=1}^{m}(X_i-EX_i)}{b_m}\right|\ge \frac{\epsilon}{2}\log n\right\}$$

$$+P\left\{\max_{m+1\le k\le n}\left|\frac{\sum_{i=m+1}^{k}(X_i-EX_i)}{b_k}\right|\ge \frac{\epsilon}{2}\log n\right\}$$

$$\le P\left\{\max_{1\le k\le m}\left|\frac{\sum_{i=1}^{k}(X_i-EX_i)}{b_m}\right|\ge \frac{\epsilon}{2}\log n\right\}$$

$$+P\left\{\max_{m+1\le k\le m}\left|\frac{\sum_{i=m+1}^{k}(X_i-EX_i)}{b_k}\right|\ge \frac{\epsilon}{2}\log n\right\}$$

$$\le C(\epsilon\log n)^{-2}((\log n/\log 3)+2)^2\left(\sum_{j=m+1}^{n}\frac{1+EX_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{1+EX_j^2}{b_m^2}\right).$$

## 4.3 Applications of the Hájek-Rényi inequality to the strong laws of large numbers for AQSI sequence

The first result in this section provides the strong law of large numbers for a sequence of AQSI random variables.

**Theorem 4.3.1** *Let $\{X_n, n\ge 1\}$ be a sequence of AQSI random variables with $\sigma_n^2 = EX_n^2<\infty, n\ge 1, \sum_{m=1}^{\infty}q(m)<\infty$, and satisfying (4.9) and (4.10). Let $\{b_n, n\ge 1\}$ be a positive sequence of nondecreasing real numbers. If*

$$\sum_{n=1}^{\infty}\frac{1+\sigma_n^2}{b_n^2}<\infty, \tag{4.13}$$

*holds, then*

$$(b_n \log n)^{-1} \sum_{i=1}^{n} (X_i - EX_i) \to 0 \quad a.s. \ as \quad n \to \infty.$$

**Proof.** Using the Hájek-Rényi inequality presented in the previous section, we have

$$P\left( \max_{m \le k \le n} \left| \frac{\sum_{i=1}^{k} (X_i - EX_i)}{b_n} \right| \ge \epsilon \log n \right)$$
$$\le C(\epsilon \log n)^{-2}((\log n / \log 3) + 2)^2 \left( \sum_{j=m+1}^{n} \frac{1 + \sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{1 + \sigma_j^2}{b_m^2} \right).$$

But

$$P\left( \sup_n \left| \frac{\sum_{i=1}^{n} (X_i - EX_i)}{b_n} \right| \ge \epsilon \log n \right) \le \lim_{n \to \infty} P\left( \max_{m \le k \le n} \left| \frac{\sum_{i=1}^{k} (X_i - EX_i)}{b_k} \right| \ge \epsilon \log n \right)$$
$$\le C \lim_{n \to \infty} (\epsilon \log n)^{-2}((\log n / \log 3) + 2)^2 \left( \sum_{i=m+1}^{n} \frac{1 + \sigma_i^2}{b_i^2} + \sum_{j=1}^{m} \frac{1 + \sigma_j^2}{b_m^2} \right)$$
$$\le C \left( \sum_{j=m+1}^{\infty} \frac{1 + \sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{1 + \sigma_j^2}{b_m^2} \right).$$

Hence, by the Kronecker lemma and (4.13), we obtain

$$\lim_{n \to \infty} P\left( \sup_n \frac{\sum_{i=1}^{n} (X_i - EX_i)}{b_n} \ge \epsilon \log n \right) = 0,$$

which completes the proof.

# Chapter 5

# The Hájek-Rényi Inequality for Asymptotically Almost Negatively Associated (AANA) Random Variables

The results of this chapter are based on the paper by Ko, Kim, and Lin (2005)

## 5.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. The definition of the negatively associated sequence of random variables was introduced in Chapter 2. A sequence $\{X_n, n \geq 1\}$ of random variables is called *asymptotically almost negatively associated (AANA)*, if there is a nonnegative sequence $q(m) \to 0$ such that

$$\operatorname{Cov}(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \le q(m)(\operatorname{Var}(f(X_m))\operatorname{Var}(g(X_{m+1}, \cdots, X_{m+k})))^{\frac{1}{2}}, \tag{5.1}$$

for all $m, k \ge 1$ and for all increasing continuous scalar functions $f$ and coordinatewise increasing continuous functions $g$, whenever the right-hand side of (5.1) is finite. This definition was introduced by Chandra and Ghosal (1996a,b). By using this inequality, Chandra and Ghosal (1996a) derived the Kolmogorov type maximal inequality for AANA random variables and obtained the strong law of large numbers for AANA random variables. Chandra and Ghosal (1996b) also derived the almost sure convergence of weighted averages on the AANA random variables. In this chapter, we derive the Hájek-Rényi type inequality of asymptotically almost negatively associated (AANA) random variables and apply this inequality to obtain the strong law of large numbers for weighted sums of AANA sequences.

## 5.2 The Hájek-Rényi inequality for AANA sequences

In order to prove the main result of this section, we need the following lemma.

**Lemma 5.2.1** *Let* $\{X_n, n \ge 1\}$ *be a sequence of AANA random variables with* $EX_k = 0$ *and* $\sigma_k^2 = EX_k^2 < \infty, k \ge 1$. *Suppose that there exist* $M > 1$ *and* $D > 0$

*such that for all $n \geq 1$,*

$$\left(\sum_{k=1}^{n} \sigma_k^{M/(M-1)}\right)^{1-1/M} \leq D\left(\sum_{k=1}^{n} \sigma_k^2\right)^{1/2}. \tag{5.2}$$

*Let $A = D(\sum_{m=1}^{n-1} q^M(m))^{1/M}$. Then*

$$E\left(\max_{1\leq k\leq n} \sum_{i=1}^{k} X_i\right)^2 \leq (A + (1 + A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2, \tag{5.3}$$

*and*

$$E\left(\sum_{k=1}^{n} X_k\right)^2 \leq (A + (1 + A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{5.4}$$

**Proof**. See Ko, Kim, and Lin (2005).

Now, we can present the main result of this section as follows.

**Theorem 5.2.1** *Let $\{a_n, n \geq 1\}$ be a positive sequence of real numbers and $\{b_n, n \geq 1\}$ a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_n = 0$ and $\sigma_n^2 = EX_n^2 < \infty$. Suppose that condition (5.2) is satisfied and $A$ is defined as in Lemma 5.2.1. Then*

$$P\left\{\max_{1\leq k\leq n} \left|\frac{\sum_{i=1}^{k} a_i X_i}{b_k}\right| \geq \epsilon\right\} \leq 8\epsilon^{-2}(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \frac{a_k^2 \sigma_k^2}{b_k^2}. \tag{5.5}$$

**Proof.** Without loss of generality, we assume that $b_0 = 0$. Let $T_k = \sum_{j=1}^{k} a_j X_j$, we have

$$T_k = \sum_{j=1}^{k} b_j \frac{a_j X_j}{b_j} = \sum_{j=1}^{k} \left( \sum_{i=1}^{j} (b_i - b_{i-1}) \frac{a_j X_j}{b_j} \right) = \sum_{i=1}^{k} (b_i - b_{i-1}) \sum_{i \le j \le k} \frac{a_j X_j}{b_j}.$$

Note that $(1/b_k) \sum_{j=1}^{k} (b_j - b_{j-1}) = 1$. Hence, the event

$$\left\{ \left| \frac{T_k}{b_k} \right| \ge \epsilon \right\} \subset \left\{ \max_{1 \le i \le k} \left| \sum_{i \le j \le k} \frac{a_j X_j}{b_j} \right| \ge \epsilon \right\}.$$

Therefore,

$$\begin{aligned} \left\{ \max_{1 \le k \le n} \left| \frac{S_k}{b_k} \right| \ge \epsilon \right\} &\subset \left\{ \max_{1 \le k \le n} \max_{1 \le i \le k} \sum_{i \le j \le k} \left| \frac{a_j X_j}{b_j} \right| \ge \epsilon \right\} \\ &= \left\{ \max_{1 \le i \le k \le n} \left| \sum_{j \le k} \frac{a_j X_j}{b_j} - \sum_{j < i} \frac{a_j X_j}{b_j} \right| \ge \epsilon \right\} \\ &\subset \left\{ \max_{1 \le i \le n} \left| \sum_{j=1}^{i} \frac{a_j X_j}{b_j} \right| \ge \frac{\epsilon}{2} \right\}, \end{aligned}$$

which completes the proof.

From this theorem, we can get the following generalized Hájek-Rényi inequality.

**Theorem 5.2.2** *Let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers and $\{b_n, n \ge 1\}$ a sequence of nondecreasing positive real numbers. Let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $\sigma_k^2 = EX_k^2 < \infty$. Suppose that condition*

*(5.2) is satisfied and $A$ is defined as in Lemma 5.2.1. Then for any $\epsilon > 0$ and any positive integer $m < n$,*

$$P\{\max_{m\leq k\leq n} |\frac{\sum_{i=1}^{k} a_i X_i}{b_k}| \geq \epsilon\} \leq 8\epsilon^{-2}(A + (1 + A^2)^{\frac{1}{2}})^2 (\sum_{j=m+1}^{n} \frac{a_j^2 \sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{a_j^2 \sigma_j^2}{b_m^2}). \quad (5.6)$$

**Proof.** See Ko, Kim, and Lin (2005).

**Theorem 5.2.3** *Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers and $\{b_n, n \geq 1\}$ a sequence of nondecreasing positive real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $\sigma_k^2 = EX_k^2 < \infty$. Denote $T_n = \sum_{i=1}^{n} a_i X_i, n \geq 1$. Suppose that condition (5.2) is satisfied and that*

$$\sum_{n=1}^{\infty} (\frac{a_n}{b_n})^2 \sigma_n^2 < \infty, \quad (5.7)$$

*and*

$$\left( \sum_{k=1}^{\infty} q^M(k) \right)^{1/M} < \infty \text{ for } M \geq 2, \quad (5.8)$$

*hold. Then*

*(1) for any $0 < r < 2, E\sup_n(|T_n|/b_n)^r < \infty$, and*

*(2) $0 < b_n \uparrow \infty$ implies $T_n/b_n \to 0$ a.s. as $n \to \infty$.*

**Proof.** Let $B = D(\sum_{k=1}^{\infty} q^M(k))^{1/M}$.

(1): Note that, for any $0 < r < 2$,

$$E\sup_n(\frac{|T_n|}{b_n})^r < \infty \Longleftrightarrow \int_1^\infty P\left\{\sup_n \frac{|T_n|}{b_n} > t^{\frac{1}{r}}\right\}\ dt < \infty.$$

By Theorem 5.2.1 above, it follows from (5.7) and (5.8) that

$$\begin{aligned}\int_1^\infty P\left\{\sup_n \frac{|T_n|}{b_n} > t^{\frac{1}{r}}\right\}\ dt &\le 2\int_1^\infty t^{-\frac{2}{r}}\left(B+(1+B^2)^{\frac{1}{2}}\right)^2\sum_{n=1}^\infty\left(\frac{a_n}{b_n}\right)^2\sigma_n^2 dt\\ &= 2\left(B+(1+B^2)^{\frac{1}{2}}\right)^2\sum_{n=1}^\infty\left(\frac{a_n}{b_n}\right)^2\sigma_n^2\int_1^\infty t^{-\frac{2}{r}}dt\\ &< \infty.\end{aligned}$$

Hence, the proof of (1) is complete.

(2): By Theorem 5.2.2 above, we have

$$P\left\{\max_{m\le k\le n}\left|\frac{\sum_{i=1}^k a_iX_i}{b_k}\right| \ge \epsilon\right\} \le 8\epsilon^{-2}\left(B+(1+B^2)^{\frac{1}{2}}\right)^2\left(\sum_{j=m+1}^n \frac{a_j^2\sigma_j^2}{b_j^2}+\sum_{j=1}^m\frac{a_j^2\sigma_j^2}{b_m^2}\right). \tag{5.9}$$

But

$$\begin{aligned}P\left\{\sup_{k\ge m}\left|\frac{\sum_{i=1}^k a_iX_i}{b_k}\right| \ge \epsilon\right\} &= \lim_{n\to\infty} P\left\{\max_{m\le j\le n}\left|\frac{\sum_{i=1}^j a_iX_i}{b_j}\right| \ge \epsilon\right\}\\ &\le 8\epsilon^{-2}\left(B+(1+B^2)^{\frac{1}{2}}\right)^2\left(\sum_{j=m+1}^\infty \frac{a_j^2\sigma_j^2}{b_j^2}+\sum_{j=1}^m\frac{a_j^2\sigma_j^2}{b_m^2}\right)\\ &< \infty,\end{aligned}$$

by (5.7) and (5.8). By the Kronecker lemma, it follows by (5.7) that

$$\sum_{j=1}^{m} \frac{a_j^2 \sigma_j^2}{b_m^2} \to 0 \text{ as } m \to \infty \tag{5.10}$$

Hence, combining (5.7)–(5.10) yields

$$\lim_{n\to\infty} P\left\{\sup_{k\geq n}\left|\frac{\sum_{i=1}^{k} a_i X_i}{b_k}\right| \geq \epsilon\right\} = 0,$$

which completes the proof of (2).

## 5.3 The Marcinkiewicz-Zygmund strong law of large numbers for AANA random variables

The following theorem gives us the Marcinkiewicz-Zygmund strong law of large numbers for AANA random variables.

**Theorem 5.3.1** *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed AANA random variables. Assume that conditions (5.2) and (5.8) are satisfied.*

*(1) If $E|X_1|^t < \infty$ for some $0 < t < 1$, then $\sum_{i=1}^{n} X_i / n^{\frac{1}{t}} \to 0$. a.s.*

*(2) If $E|X_1|^t < \infty$ for some $1 \leq t < 2$, then $\sum_{i=1}^{n} (X_i - EX_i)/n^{\frac{1}{t}} \to 0$. a.s.*

**Proof.** Suppose that $E|X_1|^t < \infty$ for some $0 < t < 2$. Let $X_i^+ = \max(X_i, 0), X_i^- = \max(-X_i, 0)$. To prove (1), it suffices to show that

$$\frac{\sum_{i=1}^n X_i^+}{n^{1/t}} \to 0 \text{ a.s. and} \tag{5.11}$$

$$\frac{\sum_{i=1}^n X_i^-}{n^{1/t}} \to 0 \text{ a.s.} \tag{5.12}$$

To prove 2, it suffices to show that

$$\frac{\sum_{i=1}^n (X_i^+ - EX_i^+)}{n^{1/t}} \to 0 \text{ a.s. and} \tag{5.13}$$

$$\frac{\sum_{i=1}^n (X_i^- - EX_i^-)}{n^{1/t}} \to 0 \text{ a.s.} \tag{5.14}$$

Note that $\{X_i^+, i \geq 1\}$, $\{X_i^-, i \geq 1\}$ are AANA random variables. Let $Y_i = \min\{X_i^+, n^{1/t}\}$, $i = 1, \cdots, n$ so that $\{Y_i, 1 \leq i \leq n\}$ are identically distributed AANA random variables. Notice that $E|X_1|^t < \infty$ implies $\sum_{n=1}^\infty P(|X_1| > n^{1/t}) < \infty$ and, on the other hand,

$$P(Y_i \neq X_i^+) = P(X_1^+ \wedge n^{1/t} \neq X_i^+) \leq P(X_1^+ > n^{1/t}) \leq P(|X_1| > n^{1/t}).$$

Therefore,

$$P(Y_i \neq X_1^+ \text{i.o.}) = 0. \tag{5.15}$$

We will prove first

$$\frac{\sum_{i=1}^n EY_i}{n^{1/t}} \to 0 \text{ for } 0 < t < 1, \quad \text{and} \tag{5.16}$$

$$\frac{\sum_{i=1}^n (EX_i^+ - EY_i)}{n^{1/t}} \to 0 \text{ for } 1 \le t < 2. \tag{5.17}$$

We start with the proof of (5.16). Notice that

$$\begin{aligned}
\sum_{n=1}^\infty \frac{EY_n}{n^{1/t}} &= \sum_{n=1}^\infty n^{-1/t}\{EX_1^+ I(X_1^+ \le n^{1/t}) + n^{1/t} P(X_1^+ > n^{1/t})\} \\
&= \sum_{n=1}^\infty n^{-1/t} EX_1^+ I(X_1^+ \le n^{1/t}) + \sum_{n=1}^\infty P(X_1^+ > n^{1/t}) \\
&\le \sum_{n=1}^\infty n^{-1/t} \sum_{k=1}^n EX_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + \sum_{n=1}^\infty P(|X_1| > n^{1/t}) \\
&\le \sum_{k=1}^\infty EX_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) \sum_{n=k}^\infty n^{-1/t} + E|X_1|^t \\
&\le C \sum_{k=1}^\infty k^{-1(1/t)+1} EX_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le C \sum_{k=1}^\infty E(X_1^+)^t I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le CE|X_1|^t < \infty.
\end{aligned}$$

By Kronecker's lemma , (5.16) is true. Now, we prove (5.17). Since $|EX_n^+ - EY_n| \le EX_n^+ I(X_n^+ > n^{1/t}) + n^{1/t} P(X_n^+ > n^{1/t})$, it follows in a similar way as above that $\sum_{n=1}^\infty n^{-1/t} |EX_n^+ - EY_n| < \infty$. Hence, (5.17) is proved. From (5.15), (5.16) and

(5.17), it suffices to show now that

$$\frac{\sum_{i=1}^{n}(Y_i - EY_i)}{n^{1/t}} \to 0 \text{ a.s.} \tag{5.18}$$

By Theorem 5.2.3, taking $b_n = n^{1/t}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2/t} E(Y_n - EY_n)^2 \\
&\quad \le \sum_{n=1}^{\infty} n^{-2/t} EY_n^2 \\
&\quad = \sum_{n=1}^{\infty} n^{-2/t} E(X_1^+ \wedge n^{1/t})^2 \\
&\quad = \sum_{n=1}^{\infty} n^{-2/t} E(X_1^+)^2 I(X_1^+ \le n^{1/t}) + \sum_{n=1}^{\infty} P(X_1^+ > n^{1/t}) \\
&\quad \le \sum_{n=1}^{\infty} n^{-2/t} \sum_{k=1}^{n} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\quad = \sum_{n=1}^{\infty} n^{-2/t} \sum_{k=1}^{n} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) \sum_{n=k}^{\infty} n^{-2/t} + E|X_1|^t \\
&\quad \le C \sum_{k=1}^{\infty} k^{-(2/t)+1} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\quad \le C \sum_{k=1}^{\infty} k^{-(2/t)+1} k^{(2/t)-1} E(X_1^+)^t I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\quad \le CE|X_1|^t < \infty.
\end{aligned}$$

The proof is complete.

# Chapter 6

# The Hájek-Rényi Inequality and the Strong Law of Large Numbers

The results of this chapter are based on the paper by Yang, and Su (2000)

## 6.1 Introduction

Proving the strong law of large numbers can be done by determining the desired result for a particular subsequence, before taking the whole sequence and reducing the problem, so that the subsequence result is applicable. The determination of a maximal inequality for the whole sequence of cumulative sums is necessary. Previous research, in probability theory, has found numerous maximal inequalities for various classes of random variables. Therefore, individual determination may not be necessary. An alternative method of proving the strong law of large numbers is more difficult, but can be done by applying a maximal inequality for normalized sums. This inequality is referred to as the Hájek-Rényi inequality, in honour of a paper written by Hájek and Rényi (1955) which describes independent summands.

In this chapter, we prove that the Hájek-Rényi type inequality is, in fact, a consequence of an appropriate maximal inequality for cumulative sums, and also show that the latter automatically implies the strong law of large numbers.

Fazekas and Klesov (2000) showed an approach using the Hájek-Rényi type inequaltiy that allows the strong law of large numbers to be found for a sequence of random variables. This approach can be applied to sequences, which are made up of dependent random variables that have undergone normalization of their partial sums.

## 6.2 General Hájek-Rényi type maximal inequality

Let $\{S_n, n \geq 1\}$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathfrak{F}, P)$. The following theorem proved in Fazekas and Klesov (2000), seems to be the most general form of the Hájek-Rényi type inequality. We would like readers to be careful that no assumptions on the dependence structure of a sequence of random variables $\{S_n, n \geq 1\}$ are made.

**Theorem 6.2.1** *Let $\beta_1, \cdots, \beta_n$ be a nondecreasing sequence of positive numbers, $\alpha_1, \cdots, \alpha_n$ be nonnegative numbers and $\{S_n, n \geq 1\}$ be an arbitrary sequence of random variables. Let $r$ be a fixed positive number. Assume that for each $m$ with*

$1 \leq m \leq n$,

$$E[\max_{1\leq l\leq m} |S_l|]^r \leq \sum_{l=1}^{m} \alpha_l.$$

*Then*

$$E[\max_{1\leq l\leq n} |\frac{S_l}{\beta_l}|]^r \leq 4\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r}.$$

**Proof.** We can assume that $\beta_1 = 1$. Let $c = 2^{1/r}$. Consider the sets $A_i = \{k : c^i \leq \beta_k < c^{i+1}\}$, $i = 0, 1, 2, \cdots$. Denote by $i(n)$ the index of the last nonempty $A_i$. Let $k(i) = \max\{k : k \in A_i\}$, $i = 0, 1, 2, \cdots$ , if $A_i$ is nonempty, while $k(i) = k(i-1)$ if $A_i$ is empty, and let $k(-1) = 0$. Let

$$\delta_l = \sum_{j=k(l-1)+1}^{k(l)} \alpha_j, l = 0, 1, 2\cdots,$$

where $\delta_l$ is considered to be zero if $A_l$ is empty. We have

$$\begin{aligned}
E[\max_{1\leq l\leq n} |\frac{S_l}{\beta_l}|]^r &\leq \sum_{i=0}^{i(n)} E[\max_{l\in A_i} |\frac{S_l}{\beta_l}|]^r \leq \sum_{i=0}^{i(n)} c^{-ir} E[\max_{l\in A_i} |S_l|]^r \leq \sum_{i=0}^{i(n)} c^{-ir} E[\max_{k\leq k(i)} |S_k|]^r \\
&\leq \sum_{i=0}^{i(n)} c^{-ir} \sum_{k=1}^{k(i)} \alpha_k = \sum_{i=0}^{i(n)} c^{-ir} \sum_{l=0}^{i} \delta_l = \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{i(n)} c^{-ir} \leq \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{\infty} c^{-ir} \\
&\leq \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} \delta_l c^{-lr} = \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \\
&\leq \frac{c^r}{1-c^{-r}} \sum_{l=0}^{i(n)} \sum_{k=k(l-1)+1}^{k(l)} \frac{\alpha_k}{\beta_k^r} = 4\sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^r}.
\end{aligned}$$

The theorem is proved.

## 6.3 General method of establishing the strong law of large numbers

We first present a general theorem to the strong law of large numbers.

**Theorem 6.3.1** *Let $b_1, b_2, \cdots$ be a nondecreasing sequence of positive numbers with*

$$1 \le b_{2n}/b_n \le c < \infty, \tag{6.1}$$

*for some $c > 1$. Assume that*

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le k\le n} |S_k| > b_n \varepsilon) \le C < \infty. \tag{6.2}$$

*Then*

$$\lim_{n\to\infty} \frac{\max_{1\le j\le n} |S_j|}{b_n} = 0 \ a.s. \tag{6.3}$$

**Proof.** Since $b_n \le b_{2n} \le c b_n$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} P(\max_{1\le j\le 2^k} |S_j| > \varepsilon b_{2^k}) &= \sum_{k=1}^{\infty} \sum_{2^k \le n \le 2^{k+1}} (2^k)^{-1} P(\max_{1\le j\le 2^k} |S_j| > \varepsilon b_{2^k}) \\ &\le \sum_{k=1}^{\infty} \sum_{n\text{: even and } 2^k \le n < 2^{k+1}} (n/2)^{-1} P(\max_{1\le j\le n} |S_j| > \varepsilon b_{n/2}) \\ &\quad + \sum_{k=1}^{\infty} \sum_{n\text{: odd and } 2^k \le n < 2^{k+1}} (n/2)^{-1} P(\max_{1\le j\le n} |S_j| > \varepsilon b_{(n+1)/2})\end{aligned}$$

$$\le 2^{-1}\sum_{k=1}^{\infty}\sum_{n:\ \text{even and }2^k\le n<2^{k+1}}(n)^{-1}P(\max_{1\le j\le n}|S_j|>\varepsilon c^{-1}b_n)$$

$$+\,2^{-1}\sum_{k=1}^{\infty}\sum_{n:\ \text{odd and }2^k\le n<2^{k+1}}(n)^{-1}P(\max_{1\le j\le n}|S_j|>\varepsilon c^{-1}b_{n+1})$$

$$\le 2^{-1}\sum_{k=1}^{\infty}\sum_{2^k\le n<2^{k+1}}(n)^{-1}P(\max_{1\le j\le n}|S_j|>\varepsilon c^{-1}b_n)$$

$$=2^{-1}\sum_{n=2}^{\infty}(n)^{-1}P(\max_{1\le j\le n}|S_j|>\varepsilon c^{-1}b_n)<\infty.$$

Using the Borel-Cantelli lemma, this implies that $\max_{1\le j\le 2^k}|S_j/b_{2^k}|\to 0$, a.s. $k\to\infty$. Furthermore,

$$\max_{2^{k-1}<n\le 2^k}\max_{1\le j\le n}|S_j/b_n|\le \max_{2^{k-1}<n\le 2^k}\max_{1\le j\le 2^k}|S_j/b_{2^{k-1}}|\le \max_{2^{k-1}<n\le 2^k}\max_{1\le j\le 2^k}|S_j/b_{2^k/2}|$$

$$\le c^{-1}\max_{2^{k-1}<n\le 2^k}\max_{1\le j\le 2^k}|S_j/b_{2^k}|=c^{-1}\max_{1\le j\le 2^k}|S_j/b_{2^k}|\to 0\text{ a.s.}$$

as $k\to\infty$. Using the sub-sequence method, we obtain that $\max_{1\le j\le n}|S_j/b_n|\to 0$ a.s. $n\to\infty$. If $b_n$ is geometrically increasing, such as $b_n=\rho^n$ for some $\rho>1$, then $b_{2n}/b_n$ is increasing and unbounded. In this case, (6.3) holds under a very weak condition. We have a general result as follows.

**Theorem 6.3.2** *Assume that* $\sup_{j\ge 1}E|X_j|^r<\infty$ *for some* $0<r<1$. *If* $b_{2n}/b_n$ *is increasing and unbounded, then (6.3) holds.*

**Proof.** In fact, from the fact that $b_{2n}/b_n$ is increasing and unbounded, we have that for any given $M > 0$, $b_{2n}/b_n \geq M$ for sufficiently large $n$. Without loss of generality, assume that $b_{2^k}/b_{2^{k-1}} \geq M$ for all $k > 1$. Hence, $b_{2^k} \geq M b_{2^{k-1}} \geq \cdots \geq M^k b_1$. Take $M = 5^{1/r}$,

$$\begin{aligned}\sum_{n=1}^{\infty} P(\max_{1\leq j\leq n}|S_j| > \varepsilon b_n) &\leq C\sum_{n=1}^{\infty} b_n^{-r} E\max_{1\leq j\leq n}|S_j|^r \leq C\sum_{n=1}^{\infty} b_n^{-r}\sum_{j=1}^{\infty} E|X_j|^r \leq C\sum_{n=1}^{\infty} n b_n^{-r} \\ &= C\sum_{k=1}^{\infty}\sum_{2^{k-1}\leq n<2^k} n b_n^{-r} \leq C\sum_{k=1}^{\infty}\sum_{2^{k-1}\leq n<2^k} 2^k b_{2^{k-1}}^{-r} = C\sum_{k=1}^{\infty} 2^{2(k-1)} b_{2^{k-1}}^{-r} \\ &\leq C\sum_{k=1}^{\infty} 2^{2(k-1)} M^{-r(k-1)} = C\sum_{k=1}^{\infty} (4/M^r)^{k-1} < \infty.\end{aligned}$$

Hence, (6.3) holds.

## 6.4 The strong law of large numbers for positively associated sequence

The concept of the associated random variables was introduced by Esary *et al.* (1967) in the following way. Consider the finite family of random variables $\{X_i, 1 \leq i \leq n\}$, with finite second moments. We call $\{X_i, 1 \leq i \leq n\}$ *associated* if $\mathrm{Cov}(f, g) > 0$ for any real coordinate-wise nondecreasing scalar functions $f = f(X_1, \ldots, X_n)$ and $g = g(X_1, \ldots, X_n)$ on $\mathbf{R}^n$. An infinite family of random variables $\{X_i, i \geq 1\}$ is *associated* if every finite subfamily is associated.

**Theorem 6.4.1** *Let $\{X_j : j \geq 1\}$ be a sequence of positively associated random variables with*

$$\sum_{i=1}^{\infty} u^{1/2}(2^i) < \infty. \tag{6.4}$$

*Let $\varphi$: $R \to R^+$ be an even and nondecreasing on $[0,\infty)$ function with $\lim_{x\to\infty}\varphi(x) = \infty$, and such that*

*(1) $\varphi(x)/x$ is nonincreasing,*

*(2) $\varphi(x)/x$ is nondecreasing and $\varphi(x)/x^2$ is nonincreasing as $x \to \infty$ and $EX_j = 0$.*

*Assume that $b_1, b_2, \cdots$ is a nondecreasing sequence of positive numbers satisfying*

$$1 \leq b_{2n}/b_n \leq c < \infty, \forall n \geq 1, \tag{6.5}$$

$$\sum_{j=1}^{\infty} P(|X_j| > b_j) < \infty, \tag{6.6}$$

$$\sum_{j=1}^{\infty} \frac{E\varphi(X_j)I(|X_j| \leq b_j)}{\varphi(b_j)} < \infty, \tag{6.7}$$

$$\sum_{n=1}^{\infty} b_n^{-2} \max_{1\leq j\leq n} b_j^2 P(|X_j| > b_j) < \infty, \tag{6.8}$$

$$\sum_{n=1}^{\infty} b_n^{-2} \max_{1\leq j\leq n} \frac{b_j^2 \varphi(X_j)I(|X_j| \leq b_j)}{\varphi(b_j)} < \infty. \tag{6.9}$$

*Then*

$$S_n/b_n \to 0, \ \ a.s. \tag{6.10}$$

**Proof.** Let $Z_j := X_j I(|X_j| \le b_j) - b_j I(X_j < -b_j) + b_j I(X_j > b_j)$. we first show that

$$b_n^{-1} |E \sum_{j=1}^{n} Z_j| \to 0. \tag{6.11}$$

If $\varphi(x)$ satisfies the condition $\varphi(x)/x$ is nonincreasing, then $\frac{\varphi(b_j)}{b_j} \le \frac{\varphi(|X_j|)}{|X_j|}$ as $|X_j| \le b_j$, hence $\frac{|X_j|}{b_j} \le \frac{\varphi(X_j)}{\varphi(b_j)}$. Therefore,

$$\begin{aligned}
|EZ_j| &\le b_j E(|X_j|/b_j) I(|X_j| \le b_j) + b_j P(|X_j| > b_j) \\
&\le b_j E(\varphi(|X_j|)/\varphi(b_j)) I(|X_j| \le b_j) + b_j P(|X_j| > b_j) \\
&\le b_j E\varphi(X_j) I(|X_j| \le b_j)/\varphi(b_j) + b_j P(|X_j| > b_j).
\end{aligned}$$

If $\varphi(x)$ satisfies the condition $\varphi(x)/x$ is nondecreasing, then $\frac{\varphi(b_j)}{b_j} \le \frac{\varphi(|X_j|)}{|X_j|}$ as $|X_j| > b_j$. Therefore, $\frac{|X_j|}{b_j} \le \frac{\varphi(X_j)}{\varphi(b_j)}$. Note that $EX_j = 0$, we get

$$\begin{aligned}
|EZ_j| &= |EX_j I(|X_j| \le b_j)| + b_j P(|X_j| > b_j) = |EX_j I(|X_j| > b_j)| + b_j P(|X_j| > b_j) \\
&\le b_j E(|X_j|/b_j) I(|X_j| > b_j) + b_j P(|X_j| > b_j) \\
&\le b_j E(\varphi(|X_j|)/\varphi(b_j)) I(|X_j| \ge b_j) + b_j P(|X_j| > b_j).
\end{aligned}$$

We have that

$$|EZ_j| \le b_j E\varphi(X_j) I(|X_j| \le b_j)/\varphi(b_j) + b_j P(|X_j| > b_j). \tag{6.12}$$

From (6.6), (6.7), and(6.12), we get $\sum_{j=1}^{\infty} |EZ_j|/b_j < \infty$. Thus, $b_n^{-1} \sum_{j=1}^{n} |EZ_j|/b_j \to 0$ by Kronecker's lemma. This implies (6.11). On the other hand, from (6.6)

$$\sum_{j=1}^{\infty} P(X_j \neq Z_j) = \sum_{j=1}^{\infty} P(|X_j| > b_j) < \infty. \tag{6.13}$$

Then, (6.11) and (6.13) imply that it is sufficient to prove

$$b_n^{-1} \sum_{j=1}^{n} (Z_j - EZ_j) \to 0, \text{ a.s.} \tag{6.14}$$

Now, we say it is true that

$$\frac{X_j^2}{b_j^2} \leq \frac{\varphi(X_j)}{\varphi(b_j)} \text{ for } |X_j| \leq b_j. \tag{6.15}$$

In fact, if $\varphi(x)$ satisfies the condition 2, then $\varphi(x)/x^2$ is nonincreasing. So

$$\frac{\varphi(b_j)}{b_j^2} \leq \frac{|\varphi(X_j)|}{X_j^2} = \frac{\varphi(X_j)}{X_j^2},$$

implies (6.15). If $\varphi(x)$ satisfies the condition $\varphi(x)/x$ is nonincreasing, then $\frac{\varphi(b_j)}{b_j} \leq \frac{\varphi(|X_j|)}{|X_j|}$ for $|X_j| \leq b_j$, implies that $\frac{X_j^2}{b_j^2} \leq \frac{\varphi^2(|X_j|)}{\varphi^2(b_j)}$. Note that $0 < \frac{\varphi(|X_j|)}{b_j} \leq 1$ from that $\varphi(x)$ is nondecreasing on $(0, \infty)$. We have

$$\frac{X_j^2}{b_j^2} \leq \frac{\varphi^2(|X_j|)}{\varphi^2(b_j)} \leq \frac{\varphi(|X_j|)}{\varphi(b_j)} = \frac{\varphi(X_j)}{\varphi(b_j)},$$

yields (6.15). By (6.16), we know that

$$\begin{aligned} EZ_j^2 &= EX_j^2 I(|X_j| \le b_j)| + b_j^2 P(|X_j| > b_j) \\ &= b_j^2 E(X_j^2/b_j^2) I(|X_j| \le b_j)| + b_j^2 P(|X_j| > b_j) \\ &\le b_j^2 E(\varphi(X_j)/\varphi(b_j)) I(|X_j| \le b_j) + b_j^2 P(|X_j| > b_j). \end{aligned}$$

Thus, using (6.8) and (6.9), we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le k\le n} |\sum_{j=1}^{k} (Z_j - EZ_j)| > b_n \varepsilon) &\le \varepsilon^{-2} \sum_{n=1}^{\infty} n^{-1} b_n^{-2} E \max_{1\le k\le n} |\sum_{j=1}^{k} (Z_j - EZ_j)|^2 \\ &\le C \sum_{n=1}^{\infty} b_n^{-2} \max_{1\le j\le n} EZ_j^2 \\ &\le C \sum_{n=1}^{\infty} b_n^{-2} \max_{1\le j\le n} b_j^2 \{E(\varphi(X_j)/\varphi(b_j)) I(|X_j| \le b_j) + P(|X_j| > b_j)\} < \infty. \end{aligned}$$

By Theorem 6.3.1, this yields the desired result.

**Corollary 6.4.1** *Assume that $\{X_j, j \ge 1\}$ is a sequence of positively associated random variables with zero means and $\sup_{j\ge 1} E|X_j|^p < \infty$ for some $1 \le p \le 2$, and satisfying (6.4). Then for any $\delta > 1$,*

$$S_n/(n \log n (\log\log n)^{\delta})^{1/p} \to 0, \ \ a.s. \tag{6.16}$$

**Proof.** Take $\varphi(x) = |x|^p$ and $b_n = (n \log n (\log\log n)^\delta)^{1/p}$ in Theorem 6.4.1. Obviously, $1 \leq b_{2n}/b_n \leq 3$, (6.6) and (6.7) holds. Furthermore,

$$\sum_{n=1}^{\infty} b_n^{-2} \max_{1\leq j\leq n} \frac{b_j^2 E\varphi(X_j)}{\varphi(b_j)} \leq C \sum_{n=1}^{\infty} (n \log n (\log\log n)^\delta)^{-2/p} \max_{1\leq j\leq n} (j \log j (\log\log j)^\delta)^{2/p-1}$$
$$\leq C \sum_{n=1}^{\infty} (n \log n (\log\log n)^\delta)^{-1} < \infty.$$

This implies (6.8) and (6.9).

**Corollary 6.4.2** *Assume that* $\{X_j, j \geq 1\}$ *is a sequence of positively associated random variables with identical distribution, zero means and* $E|X_1|^p < \infty$ *for some* $1 \leq p < 2$*, and satisfying (6.4). Then* $S_n/n^{1/p} \to 0$ *a.s.*

**Proof.** Take $\varphi(x) = |x|^r$ where $p < r < 2$ and $b_n = n^{1/p}$. It is easy to get that $1 \leq b_{2n}/b_n \leq 2$ and (6.6). First, by the fact that $\sum_{j=n}^{\infty} j^{-1-\delta} \leq Cn^{-\delta}$ for any given $\delta > 0$ and $E|X_1|^p < \infty$, we have

$$\sum_{j=1}^{\infty} \frac{E\varphi(X_j) I(|X_j| \leq b_j)}{\varphi(b_j)} = \sum_{j=1}^{\infty} j^{-r/p} E|X_1|^r I(|X_1| \leq j^{1/p})$$
$$= \sum_{j=1}^{\infty} j^{-r/p} \sum_{n=1}^{j} E|X_1|^r I((n-1)^{1/p} < |X_1| \leq n^{1/p})$$
$$\leq \sum_{n=1}^{\infty} n^{r/p-1} E|X_1|^p I((n-1)^{1/p} < |X_1| \leq n^{1/p}) \sum_{j=n}^{\infty} j^{-r/p}$$
$$\leq C \sum_{n=1}^{\infty} E|X_1|^r I((n-1)^{1/p} < |X_1| \leq n^{1/p}) \leq CE|X_1|^p < \infty.$$

This gives us (6.7). Next, since $\sum_{j=1}^{\infty} P(|X_1| > j^{1/p}) < \infty$, we can choose a sequence of positive numbers $\{q(j) : j \geq 1\}$ with $\sum_{j=1}^{\infty} j^{-1}q(j) < \infty$ and $j^{2/p-1}q(j) \uparrow \infty$, such that $P(|X_1| > b_j) \leq Cj^{-1}q(j)$ for all $j \geq 1$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n^{-2} \max_{1\leq j\leq n} b_j^2 P(|X_j| > b_j) &= \sum_{n=1}^{\infty} n^{-2/p} \max_{1\leq j\leq n} j^{2/p} P(|X_1| > j^{1/p}) \\ &\leq C\sum_{n=1}^{\infty} n^{-2/p} \max_{1\leq j\leq n} j^{2/p-1}q(j) \leq C\sum_{n=1}^{\infty} n^{-2/p}.n^{2/p-1}q(n) \\ &= C\sum_{n=1}^{\infty} n^{-1}q(n) < \infty.\end{aligned}$$

This gives us (6.8). Finally,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n^{-2} \max_{1\leq j\leq n} \frac{b_j^2 E\varphi(X_j)I(|X_j| \leq b_j)}{\varphi(b_j)} &= \sum_{n=1}^{\infty} n^{-2/p} \max_{1\leq j\leq n} j^{2/p-r/p} E|X_1|^r I(|X_1| \leq j^{1/p}) \\ &= \sum_{n=1}^{\infty} n^{-r/p} E|X_1|^r I(|X_j| \leq n^{1/p}) \\ &\leq CE|X_1|^p < \infty,\end{aligned}$$

which implies (6.9).

**Remark 6.4.1** *For the case $p = 2$, Corollary 6.4.1 implies that the convergence rate of $S_n/n$ is $n^{-1/2}(\log n)^{-1/2}(\log\log n)^{\delta/2}$. The result is similar to the iterated logarithm for independent random variables.*

# Chapter 7

# On the Growth Rate in the Fazekas-Klesov General Law of Large Numbers and Some Applications to the Weak Law of Large Numbers for Tail Series

The results of this chapter are my new contributions.

## 7.1 Introduction

Fazekas and Klesov (2000) gave a general method for obtaining the strong law of large numbers for sequences of random variables by the Hájek-Rényi type inequality.

This general approach, in proving the strong law of large numbers, suggests to directly use a maximal inequality (the Hájek-Rényi inequality), for a sequence of normed partial sums of dependent random variables. Under the same conditions, as found in Fazekas and Klesov (2000), Hu and Hu (2006) discovered the method for obtaining the strong growth rate for sums of random variables. Although the proof of Hu and Hu (2006) owes much to that of Fazekas and Klesov (2000), their result is sharper.

In this chapter, we find a new method for obtaining the strong growth rate for sums of random variables through using the approach of Fazekas and Klesov (2000). It allows us to generalize and sharpen the method of Hu and Hu (2006). Our method can be applied to almost all cases of the dependence structure considered in Hu and Hu (2006), and we can obtain better results. Additionally, the approach of using the Hájek-Rényi type maximal inequality to prove limit theorems, is also applied to the weak law of large numbers for tail series.

The notation, to be used for this chapter, will now be provided. Let $\{X_n, n \geq 1\}$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The partial sums of the random variables are $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ and $S_0 = 0$. Let $\varphi(x)$ be a positive function satisfying

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty \text{ and } 0 < \varphi(x) \uparrow \infty \text{ on } [c, \infty) \text{ for some } c > 0. \tag{7.1}$$

## 7.2 Main results

The following lemma generalizes Dini's theorem for scalar series.

**Lemma 7.2.1** *Let $a_1, a_2, \cdots$ be a sequence of nonnegative real numbers such that $a_n > 0$ for infinitely many $n$. Let $v_n = \sum_{i=n}^{\infty} a_i$ for $n \geq 1$. Let $\varphi(x)$ be a positive function satisfying (7.1). If $\sum_{n=1}^{\infty} a_n < \infty$, then $\sum_{n=1}^{\infty} a_n \varphi(1/v_n) < \infty$.*

**Proof.** Without loss of generality, we may assume that $c = 1$ and $v_1 \leq 1$. For each $k \geq 0$, define $n_k$ by $n_k = \min\{n : v_n \leq 2^{-k}\}$. It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \varphi(1/v_n) &= \sum_{k=0}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} a_n \varphi(1/v_n) \leq \sum_{k=0}^{\infty} \varphi(1/v_{n_{k+1}-1}) \sum_{n=n_k}^{n_{k+1}-1} a_n \\ &\leq \sum_{k=0}^{\infty} \varphi(1/v_{n_{k+1}-1}) v_{n_k} \leq \sum_{k=0}^{\infty} \frac{\varphi(2^{k+1})}{2^k}.\end{aligned}$$

Note that $\sum_{n=1}^{\infty} \varphi(n)/n^2 < \infty$ is equivalent to $\sum_{k=0}^{\infty} \varphi(2^k)/2^k < \infty$, since

$$\sum_{k=0}^{\infty} \frac{\varphi(2^k)}{2^k} \leq 4 \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{\varphi(n)}{n^2} = 4 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} \leq 4 \sum_{k=0}^{\infty} \frac{\varphi(2^{k+1})}{2^k}.$$

The result follows (7.1).

It is easy to find examples of functions $\varphi(x)$ that satisfy (7.1). Such functions are $|x|^\delta$ or $|x|^\delta (\log |x|)^\alpha$, where $0 < \delta < 1$ and $\alpha$ is any real number.

The following lemma is due to Fazekas and Klesov (2000).

**Lemma 7.2.2** *Let $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive numbers and $\{\alpha_n, n \geq 1\}$ be a sequence of nonnegative real numbers such that $\alpha_n > 0$ for infinitely many $n$. Let $r$ and $C$ be fixed positive numbers. Assume that for each $n \geq 1$*

$$E\Big(\max_{1\leq i\leq n} |S_i|\Big)^r \leq C\sum_{i=1}^{n}\alpha_i \quad \textit{and} \quad \sum_{n=1}^{\infty}\alpha_n b_n^{-r} < \infty.$$

*Then the strong law of large numbers obtains, that is, $\lim_{n\to\infty} S_n/b_n = 0$ a.s.*

The following theorem gives a sharper result than Theorem 2.1 of Fazekas and Klesov (2000) and Lemma 1.2 of Hu and Hu (2006).

**Theorem 7.2.1** *Assume that all conditions of Lemma 7.2.2 are satisfied. Let $\varphi(x)$ be a positive function satisfying (7.1). Let $\beta_n = \max_{1\leq i\leq n} b_i\varphi(1/v_i)^{-1/r}$ for $n \geq 1$, where $v_n = \sum_{i=n}^{\infty}\alpha_i b_i^{-r}$. Then the following statements hold:*

1. *If the sequence $\{\beta_n, n \geq 1\}$ is bounded, then $S_n/\beta_n = O(1)$ a.s.*

   *i.e. $|\frac{S_n}{\beta_n}| < C$ for all $n$.*

2. *If the sequence $\{\beta_n, n \geq 1\}$ is unbounded, then $S_n/\beta_n = o(1)$ a.s.*

   *i.e. $\lim_{n\to\infty} S_n/\beta_n = 0$ a.s.*

**Proof.** It is easy to see that $\{\beta_n\}$ is a nondecreasing sequence of positive numbers. Since $\beta_n \geq b_n\varphi(\frac{1}{v_n})^{-1/r}$, we have by Lemma 7.2.1 that

$$\sum_{n=1}^{\infty}\alpha_n\beta_n^{-r} \leq \sum_{n=1}^{\infty}\alpha_n\varphi(1/v_n)b_n^{-r} < \infty.$$

If the sequence $\{\beta_n, n \geq 1\}$ is unbounded, then $\lim_{n\to\infty} S_n/\beta_n = 0$ a.s. by Lemma 7.2.2. Now, assume that $\{\beta_n\}$ is bounded by some constant $D > 0$. Then

$$\sum_{n=1}^{\infty} \alpha_n \leq D^r \sum_{n=1}^{\infty} \alpha_n \beta_n^{-r} < \infty.$$

It follows by the monotone convergence theorem that

$$E\Big(\sup_{n\geq 1} |S_n|\Big)^r = \lim_{n\to\infty} E\Big(\max_{1\leq i\leq n} |S_i|\Big)^r \leq C \sum_{n=1}^{\infty} \alpha_n < \infty.$$

Hence, $\sup_{n\geq 1} |S_n| < \infty$ a.s.. Since $0 < \beta_1 \leq \beta_n$ for all $n \geq 1$, we have $S_n/\beta_n = O(1)$ a.s.

**Remark 7.2.1** *For each case of $S_n/\beta_n = O(1)$ a.s. or $S_n/\beta_n = o(1)$ a.s., it can be easily obtained that $\lim_{n\to\infty} S_n/b_n = 0$ a.s., since $\lim_{n\to\infty} \beta_n/b_n = 0$.*

**Remark 7.2.2** *For the special case of $\varphi(x) = |x|^\delta \ \ (0 < \delta < 1)$, in Lemma 1.2 of Hu and Hu (2006) is proved that $S_n/\beta_n = O(1)$ a.s. under the same conditions as found in Theorem 7.2.1. We can safely state that Theorem 7.2.1 extends and sharpens the result of Hu and Hu (2006).*

It is interesting to investigate the cases, when the sequence $\{\beta_n, n \geq 1\}$ is unbounded. But, first, we derive a useful condition for $\varphi(x)$.

**Lemma 7.2.3** *If $\varphi(x)$ is a positive function satisfying (7.1), then $\varphi(n)/n \to 0$ as $n \to \infty$.*

**Proof.** Without loss of generality, we may assume that $\varphi(x)$ is a nondecreasing on $[1, \infty)$. According to the proof of Lemma 7.2.1, we have that $\sum_{k=0}^{\infty} \varphi(2^k)/2^k < \infty$ and hence, $\lim_{k\to\infty} \varphi(2^k)/2^k = 0$. For $2^k \le n < 2^{k+1}$,

$$\frac{\varphi(2^k)}{2^{k+1}} \le \frac{\varphi(n)}{n} \le \frac{\varphi(2^{k+1})}{2^k}.$$

Therefore, we have that $\lim_{n\to\infty} \varphi(n)/n = 0$.

The following lemma shows that $\{\beta_n\}$ defined in Theorem 7.2.1 is unbounded when $\alpha_n = O(1)$.

**Lemma 7.2.4** *Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers and $\varphi(x)$ be a positive function satisfying (7.1). Assume that $\sum_{n=1}^{\infty} b_n^{-r} < \infty$ for some $r > 0$. Let $\beta_n = \max_{1\le i\le n} b_i \varphi(1/v_i)^{-1/r}$ for $n \ge 1$, where $v_n = \sum_{i=n}^{\infty} b_i^{-r}$. Then, $\{\beta_n\}$ is unbounded.*

**Proof.** For each $k \ge 0$, define $n_k$ by $n_k = \min\{n : b_n \ge 2^k\}$. Let $d_k = n_k - n_{k-1}$ for $k \ge 1$. Then, we have that

$$v_{n_k} \ge \sum_{n=n_k}^{n_{k+1}-1} b_n^{-r} \ge d_{k+1} b_{n_{k+1}-1}^{-r} \ge d_{k+1} 2^{-r(k+1)} \ge d_{k+1} 2^{-r} b_{n_k}^{-r}. \qquad (7.2)$$

It follows that

$$b_{n_k}\varphi(1/v_{n_k})^{-1/r} \geq b_{n_k}\varphi\left(\frac{2^r b_{n_k}^r}{d_{k+1}}\right)^{-1/r} = \left[\frac{2^r b_{n_k}^r/d_{k+1}}{\varphi(2^r b_{n_k}^r/d_{k+1})}\right]^{1/r}\frac{d_{k+1}^{1/r}}{2}. \tag{7.3}$$

Since $\lim_{n\to\infty} v_n = 0$, (7.2) implies that $\lim_{d_{k+1}\neq 0, k\to\infty} b_{n_k}^r/d_{k+1} = \infty$. By (7.3) and Lemma 7.2.3, we obtain that $\lim_{d_{k+1}\neq 0, k\to\infty} b_{n_k}\varphi(1/v_{n_k})^{-1/r} = \infty$. Hence, $\{\beta_n\}$ is unbounded.

As a consequence of Theorem 7.2.1 and Lemma 7.2.4, we obtain the following theorem.

**Theorem 7.2.2** *Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers and $\varphi(x)$ be a positive function satisfying (7.1). Let $r$ and $C$ be fixed positive numbers. Assume that for each $n \geq 1$*

$$E\Big(\max_{1\leq i\leq n}|S_i|\Big)^r \leq Cn \quad \text{and} \quad \sum_{n=1}^{\infty} b_n^{-r} < \infty.$$

*Let $\beta_n = \max_{1\leq i\leq n} b_i\varphi(1/v_i)^{-1/r}$ for $n \geq 1$, where $v_n = \sum_{i=n}^{\infty} b_i^{-r}$. Then $S_n/\beta_n \to 0$ a.s. as $n \to \infty$.*

## 7.3 Applications to associated random variables

By using Theorem 7.2.1 and Theorem 7.2.2, we can extend and sharpen many results from Hu and Hu (2006). For example, we will show how Theorem 3.2 of Hu and Hu (2006) for (positively) associated random variables can be improved. This improvement will consider negatively associated random variables and martingale differences. The definition of positive association is given in chapter 6.

**Theorem 7.3.1** *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with mean zero and finite variance, and $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive numbers. Assume that*

$$\sum_{n=1}^{\infty} \frac{ES_n^2 - ES_{n-1}^2}{b_n^2} < \infty.$$

*Let $\beta_n = \max_{1 \leq i \leq n} b_i \varphi(1/v_i)^{-1/2}$ for $n \geq 1$, where $v_n = \sum_{i=n}^{\infty}(ES_i^2 - ES_{i-1}^2)/b_i^2$. Then the following statements hold:*

1. *If the sequence $\{\beta_n, n \geq 1\}$ is bounded, then $S_n/\beta_n = O(1)$ a.s.*

   *i.e. $|\frac{S_n}{\beta_n}| < C$ for all $n$.*

2. *If the sequence $\{\beta_n, n \geq 1\}$ is unbounded, then $S_n/\beta_n = o(1)$ a.s.*

   *i.e. $\lim_{n\to\infty} S_n/\beta_n = 0$ a.s.*

**Proof.** The proof is similar to that of Theorem 3.2 in Hu and Hu (2006). Since $\{X_n\}$ is a sequence of associated random variables, $\{-X_n\}$ is also a sequence of associated random variables. From Theorem 2 of Newman and Wright (1981), we have

$$E\Big(\max_{1\leq i\leq n} S_i^2\Big) \leq E\Big(\max_{1\leq i\leq n} S_i\Big)^2 + E\Big(\max_{1\leq i\leq n}(-S_i)\Big)^2 \leq 2ES_n^2.$$

By the definition of associated random variables, we obtain $ES_n^2 = ES_{n-1}^2 + EX_n^2 + 2Cov(S_{n-1}, X_n) \geq ES_{n-1}^2$. Let $\alpha_n = ES_n^2 - ES_{n-1}^2$ for $n \geq 1$. Then

$$E\Big(\max_{1\leq i\leq n} S_i^2\Big) \leq 2\sum_{i=1}^{n} \alpha_i \text{ and } \sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^2} < \infty.$$

Thus, the result follows from Theorem 7.2.1.

**Remark 7.3.1** *Under the same conditions of Theorem 7.3.1 with $\varphi(x) = |x|^\delta$ $(0 < \delta < 1)$, Hu and Hu (2006) proved that $S_n/\beta_n = O(1)$ a.s., which improves Theorem 3.3 of Prakasa Rao (2002). Thus, Theorem 7.3.1 extends the result of Hu and Hu (2006). Theorem 7.3.1 especially sharpens the result of Hu and Hu (2006), when $\{\beta_n, n \geq 1\}$ is unbounded. Such an example can be obtained easily. For example, if $ES_n^2 - ES_{n-1}^2 = 1$ for all $n \geq 1$, then $\{\beta_n, n \geq 1\}$ is unbounded by Lemma 7.2.4.*

## 7.4 The weak law of large numbers for tail series

The rate of convergence for an almost surely convergent series $S_n = \sum_{j=1}^{n} X_j$ of variables $\{X_n, n \geq 1\}$ is studied in this section. More specifically, if $S_n$ converges almost surely to a random variable $S$, then the tail series $T_n \equiv S - S_{n-1} = \sum_{j=n}^{\infty} X_j$ is a well-defined sequence of random variable (referred to as the *tail series*) with $T_n \to 0$ almost surely.

The main result of this section provides conditions for

$$\sup_{k\geq n}|T_k|/b_n \to 0 \quad \text{in probability} \tag{7.4}$$

which hold for a given numerical sequence $0 < b_n = o(1)$. The result is the greatest interest when $b_n = o(1)$. Nam and Rosalsky (1996) provided an example showing *inter alia* that a.s. convergence to 0 does not necessarily hold for the expression in (7.4). Theorem 7.4.1 is a very general result and we will see that some previously obtained results are immediate corollaries of it. In Theorem 7.4.1, a condition is imposed, in general, on the joint distributions of the random variables $\{X_n, n \geq 1\}$. However, in corollary, $\{X_n, n \geq 1\}$ is a martingale difference sequence of random variables. Certainly, the result is true when $\{X_n, n \geq 1\}$ is a sequence of independent random variables. In corollary, the moment condition on the $|X_n|$, and the limiting behavior of $b_n$, are imposed.

**Theorem 7.4.1** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, $\{b_n, n \geq 1\}$ be a sequence of nonnegative numbers, $\{\alpha_n, n \geq 1\}$ be a sequence of positive numbers, and $r > 0$. Let $\sum_{j=1}^{\infty} \alpha_j < \infty$. If for all natural numbers $n < m$,*

$$E(\max_{n\leq k\leq m}|\sum_{j=n}^{k} X_j|)^r \leq \sum_{j=n}^{m} \alpha_j,$$

*then the series* $\sum_{n=1}^{\infty} X_n$ *converges a.s. and the tail series* $\{T_n = \sum_{j=n}^{\infty} X_j, n \geq 1\}$ *is a well-defined sequence of random variables. Next, if* $\sum_{j=n}^{\infty} \alpha_j = o(b_n^r)$ *as* $n \to \infty$*, then the tail series obeys the limit law*

$$\frac{\sup_{k\geq n}|T_k|}{b_n} \to 0 \text{ in probability.}$$

**Proof.** For arbitrary $\varepsilon > 0$ and $n \geq 1$, the Markov inequality implies

$$P\{\sup_{m>n} |\sum_{j=1}^{m} X_j - \sum_{j=1}^{n} X_j| > \varepsilon\} \leq \varepsilon^{-r} E \sup_{m>n} |\sum_{j=1}^{m} X_j - \sum_{j=1}^{n} X_j|^r$$
$$= \varepsilon^{-r} \lim_{m\to\infty} E \max_{n+1\leq k\leq m} |\sum_{j=n+1}^{k} X_j|^r \text{ (by the monotone convergence theorem)}$$
$$\leq \varepsilon^{-r} \lim_{m\to\infty} \sum_{j=n}^{m} \alpha_j = o(1).$$

Then by Corollary 3.3.4 of Chow and Teicher (1997), $\sum_{n=1}^{\infty} X_n$ converges a.s. Thus, the tail series $\{T_n = \sum_{j=n}^{\infty} X_j, n \geq 1\}$ is a well-defined sequence of random variables. Next, for arbitrary $\varepsilon > 0$,

$$P\left\{\frac{\sup_{k\geq n}|T_k|}{b_n} > \varepsilon\right\} \leq (\varepsilon b_n)^{-r} E \sup_{k\geq n} |T_k|^r \text{ (by the Markov inequality)}$$
$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E \max_{n\leq k\leq N} |\lim_{m\to\infty} \sum_{j=k}^{m} X_j|^r \text{ (by the monotone convergence theorem)}$$
$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E \max_{n\leq k\leq N} \lim_{m\to\infty} |\sum_{j=k}^{m} X_j|^r$$

$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E \lim_{m\to\infty} \max_{n\le k\le N} |\sum_{j=k}^{m} X_j|^r$$

$$\le (\varepsilon b_n)^{-r} \lim_{N\to\infty} \liminf_{m\to\infty} E \max_{n\le k\le N} |\sum_{j=k}^{m} X_j|^r \text{ (by Fatou's lemma)}$$

$$\le (\varepsilon b_n)^{-r} \liminf_{m\to\infty} E \max_{n\le k\le m} |\sum_{j=k}^{m} X_j|^r$$

$$\le (\varepsilon b_n)^{-r} \liminf_{m\to\infty} \sum_{j=n}^{m} \alpha_j = o(1).$$

The conclusion of the theorem now follows easily.

The next corollary was obtained by Rosalsky and Rosenblatt (1998). We present a simplified proof for this corollary.

**Corollary 7.4.1** *Let $\{S_n = \sum_{j=1}^{n} X_j, n \ge 1\}$ be a martingale and $1 \le r \le 2$. If*

$$\sum_{j=n}^{\infty} E|X_j|^r = \mathcal{O}(b_n^r),$$

*then the series $\sum_{n=1}^{\infty} X_n$ converges a.s.*

*If*

$$\sum_{j=n}^{\infty} E|X_j|^r = o(b_n^r),$$

*then the tail series obeys the limit law*

$$\frac{\sup_{k\geq n}|T_k|}{b_n} \to 0 \quad \textit{in probability}$$

**Proof.** By the Burkholder-Davis-Gundy inequality, we have that for all $m \geq n \geq 1$,

$$E \max_{n\leq k\leq m} |\sum_{j=n}^{k} X_j|^r \quad \leq \quad C \sum_{j=n}^{m} E|X_j|^r.$$

The corollary follows immediately from Theorem 7.4.1.

Certainly, Theorem 7.4.1 could be generalized on the Banach space setting. We refer the reader to the papers by Deng (1988), (1991), and (1994), Rosalsky and Volodin (2001) and (2003) for such generalizations.

# Chapter 8

# Conclusion and Future Work

Establishing strong law of large numbers for tail series (we proved only weak one in our thesis) is more difficult, but we hope that it still can be done by applying a maximal inequality for normalized sums. This inequality is referred to as the Hájek-Rényi inequality, in honour of a paper written by Hájek and Rényi (1955) describing independent summands. This thesis illustrates that the Hájek-Rényi inequality is a result of choosing the correct maximal inequality for a sequence of cumulative sums. In my future work, I will find further results on the weak and strong law of large numbers for tail series towards the application of the Hájek-Rényi inequality. Another interesting problem is to find analogs of the Hájek-Rényi inequality for different dependent structures.

# Appendix

In this section we collect some important definitions from probability theory that we frequently use throughout this thesis. Let $\Omega$ denote the sample space of outcomes.

- $\mathcal{F}$-algebra on $\Omega$

  A collection $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$-algebra on $\Omega$ if $\mathcal{F}$ is an algebra on $\Omega$ such that whenever $F_n \in \mathcal{F}$ $(n \in N)$, then $\bigcup_n F_n \in \mathcal{F}$.

- Measurable space

  A pair $(\Omega, \mathcal{F})$, where $\Omega$ is a set and $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$ is called a measurable space.

- Almost surely (a.s.)

  A statement $S$ about outcomes in $\Omega$ is said to be true almost surely (a.s.), or with probability 1 (w.p.1), if $F = \{\omega : S(\omega) \text{ is true}\} \in \mathcal{F}$ and $P(F) = 1$.

- Random variable

  Let $(\Omega, \mathcal{F})$ be our (sample space, family of events). A random variable $X : \mathbf{\Omega} \rightarrow \mathbf{R}$ is a measurable function. That is, $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$.

- Martingale

  A process $X = \{X_n, n \geq 0\}$ is called a martingale if (i) $X$ is adapted, (ii) $E(|X_n|) < \infty, \forall_n$, (iii) $E[X_n|\mathcal{F}_{n-1}] = X_{n-1}, a.s.(n \geq 1)$.

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