LIMITING OPERATIONS FOR QUANTUM RANDOM VARIABLES AND A QUANTUM MARTINGALE CONVERGENCE THEOREM

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Abstract

This thesis builds on the notion of quantum, or operator valued, probability as discussed in Farenick, Plosker, and Smith (J. Math. Phys, 2011) as well as Farenick and Kozdron (J. Math. Phys, 2012) by generalizing classical limiting results to the quantum setting. Mimicking the classical setting, we prove a continuity of quantum expectation result, which is the quantum analogue of Lebesgue's dominated convergence theorem and use it extensively to prove other limiting results. With the quantum limiting results in place, we define a quantum martingale and prove a quantum martingale convergence theorem. This quantum martingale theorem is of particular interest since it exhibits non-classical behaviour; even though the limit of the martingale exists and is unique, it is not even identifiable. In order to understand the limit it is necessary to understand the space of all mean zero quantum random variables. However, we are able to provide a partial classification of the limit through a study of the space of all mean zero quantum random variables.

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Chapter 1

Introduction

It is well known that probability plays a large role in quantum mechanics. As such there is interest in extending classical probability results to the quantum setting, where the term "quantum" can be interpreted as Hilbert space operator valued. Thus a quantum probability measure ν is a set function operating on a σ-algebra $\mathcal{F}(X)$ of subsets of a locally compact Hausdorff sample space X whose values are quantum effects: selfadjoint operators acting on a complex d-dimensional Hilbert space, where for every $E \in \mathcal{F}(X)$, the eigenvalues of the operator $\nu(E)$ are contained in the closed unit interval of $\mathbb{R}$.

The benefit of investigating quantum analogues of classical results is apparent in the ability of the quantum setting to capture behaviour not seen when using the classical results. That being said, one still wishes that the quantum structure respects the classical structure and when the Hilbert space is assumed to be one dimensional, that the quantum structure will reduce to the classical structure. In order for the quantum setting to properly reflect the classical, one must be cognizant of the consequences of using a quantum system. That is, the quantum system inherits the non-commutativity of operator algebra, and the partial order structure of the real vector space of selfadjoint operators. Indeed, suppose a and b are positive operators acting on a d-dimensional Hilbert space. If $d > 1$, a and b do not commute in general. In the case that a and b do not commute, neither ab nor ba will be a positive operator, contrary to the situation for real numbers or real valued functions. Nevertheless, both $b^{1/2}ab^{1/2}$ and $a^{1/2}ba^{1/2}$ are positive regardless of whether a and b commute where $c^{1/2}$ is used to denote the unique positive square root of a positive operator c. This fact will be integral in this work.

The goal of this thesis is to establish limiting results for quantum random variables with respect to quantum expectation and quantum conditional expectation, which were first defined in [2] and [3]. In particular, a quantum analogue of the martingale convergence theorem is proved. The non-classical properties of this quantum martingale convergence theorem are then investigated.

1.1 Outline

In chapter 2 we will introduce the objects of study as well as recall a number of important definitions and results from both [2] and [3]. In Chapter 3, we will establish the first primary results of this thesis. In particular, we will prove the continuity of quantum expectation and prove a number of useful facts about quantum conditional expectation. We also investigate the set of quantum random variables with mean zero. This collection of random variables is important for understanding results in Chapter 4.

In Chapter 4, we recall the classical notion of a martingale and collect important classical results. We then introduce the notion of a quantum martingale and prove a version of the quantum martingale convergence theorem that holds for this martingale. This particular result is of special interest since the limit exhibits non-classical behaviour. It is important to remember that although the quantum notation is designed to mimic its classical counterparts, the results are highly non-classical as a consequence of the non-commutativity of operator algebra.

Chapter 2

Basic Assumptions

In this section, we will recall definitions and define our notation as well as recall results from both [2] and [3]. In particular, we will recall basic operator algebraic facts, the definition of a positive operator valued measure and the operator valued integral defined in [3], and the definition of quantum conditional expectation, which is introduced in [2]. We will also use a non-commutative multiplication based on the change of measure defined in [2], which will greatly simplify many equations.

2.1 States and Effects

This section will serve to review the preliminary operator algebraic results as well as define the space in which we will work. For further details see [2] and [3].

Throughout this thesis, we will suppose that X is a locally compact Hausdorff space with the Borel σ-algebra denoted $\mathcal{O}(X)$ of subsets of X. We will let $\mathcal{F}(X)$ be a sub-σ-algebra of $\mathcal{O}(X)$. We will also suppose that $\mathcal{H}$ is a d-dimensional Hilbert space and the space of linear operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. It is important to remember that since $\mathcal{H}$ is finite dimensional, all linear operators on $\mathcal{H}$ will be bounded. We will take the set $\{e_1, \ldots, e_d\}$ to be an orthonormal basis for $\mathcal{H}$. The trace functional $\mathrm{Tr}(\cdot)$ on $\mathcal{B}(\mathcal{H})$ is defined for operators $z \in \mathcal{B}(\mathcal{H})$ by

$$\mathrm{Tr}(z) = \sum_{j=1}^{d} \langle ze_j, e_j \rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. If $i, j \in \{1, \ldots, d\}$, then $e_{ij} \in \mathcal{B}(\mathcal{H})$ denotes the unique operator that sends e_j to e_i and e_k to 0 if $k \neq j$. Every $z \in \mathcal{B}(\mathcal{H})$ has an adjoint $z^* \in \mathcal{B}(\mathcal{H})$, which is defined to be the unique operator for which $\langle z\xi, \eta \rangle = \langle \xi, z^*\eta \rangle$ for all ξ, $\eta \in \mathcal{H}$. An operator $z \in \mathcal{B}(\mathcal{H})$ is said to be selfadjoint if $z^* = z$.

We say that an operator $a \in \mathcal{B}(\mathcal{H})$ is positive and write $a \geq 0$ if $\langle a\eta, \eta \rangle \geq 0$ for every $\eta \in \mathcal{H}$. Moreover, we take $\mathcal{B}(\mathcal{H})_+ = \{a \in \mathcal{B}(\mathcal{H}) | a \geq 0\}$ to be the space of all positive operators acting on $\mathcal{H}$. A well known fact of operator theory is that every positive operator a has a unique positive square root, $a^{1/2}$, so that $a^{1/2}a^{1/2} = a$. An important operator-theoretic fact that we shall use often is the following; if a, $z \in \mathcal{B}(\mathcal{H})$ satisfy $az = za$ and if $a \geq 0$, then $a^{1/2}z = za^{1/2}$ as well.

If $a_1, a_2 \in \mathcal{B}(\mathcal{H})$ are selfadjoint, we write $a_1 \geq a_2$ if $a_1 - a_2 \in \mathcal{B}(\mathcal{H})_+$. This endows the real vector space of selfadjoint operators with a partial order (called the Löwner order). We also write $S(\mathcal{H})$ for the state space of $\mathcal{H}$, which is the set of all $\rho \in \mathcal{B}(\mathcal{H})_+$ with $\mathrm{Tr}(\rho) = 1$. Sometimes a state $\rho \in \mathrm{S}(\mathcal{H})$ is called a density operator. If $1 \in \mathcal{B}(\mathcal{H})$ denotes the identity operator, then $e = \frac{1}{d}1 \in \mathrm{S}(\mathcal{H})$.

We consider the set $\mathcal{B}(\mathcal{H})$ in its σ-weak topology. That is, a net $\{z_\alpha\}_\alpha \subset \mathcal{B}(\mathcal{H})$ converges to $z \in \mathcal{B}(\mathcal{H})$ if and only if the net $\{\mathrm{Tr}(\rho z_\alpha)\}_\alpha$ converges to $\mathrm{Tr}(\rho z)$ in $\mathbb{C}$ for every $\rho \in S(\mathcal{H})$. Since $\mathcal{H}$ is finite dimensional, the σ-weak topology on $\mathcal{B}(\mathcal{H})$ is the same as the norm topology of $\mathcal{B}(\mathcal{H})$, where the norm $\|z\|$ of $z \in \mathcal{B}(\mathcal{H})$ is given by $\|z\| = \sqrt{\lambda_{\max}(z^*z)}$ and where $\lambda_{\max}(z^*z)$ denotes the largest (necessarily non-negative) eigenvalue of z^*z. Therefore, the σ-algebra $\mathcal{O}(\mathcal{B}(\mathcal{H}))$ of Borel sets of $\mathcal{B}(\mathcal{H})$ is the smallest σ-algebra of subsets of $\mathcal{B}(\mathcal{H})$ that contains every set of the form $\{z \in \mathcal{B}(\mathcal{H}) : \|z - z_0\| < \varepsilon\}$, for all $z_0 \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. We define a quantum effect as a positive operator $h \in \mathcal{B}(\mathcal{H})_+$ where every eigenvalue λ of h satisfies $0 \leq \lambda \leq 1$, and let $\mathrm{Eff}(\mathcal{H})$ denote the set of all quantum effects. In fact, if $h \in \mathrm{Eff}(\mathcal{H})$, then we also have $0 \leq h \leq 1$. Note that every state $\rho \in \mathrm{S}(\mathcal{H})$ is also a quantum effect.

2.2 Positive Operator Valued Measures and Quantum Random Variables

We now recall the definition of both a positive operator valued measure and a quantum random variable. These are the operator valued counterparts to Lebesgue measure and random variables. In generalizing these classical notions to the quantum setting, some difficulties must be addressed due to the non-commutativity inherited from operator algebras.

Definition 2.2.1. A set function $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is called a positive operator valued measure on $(X, \mathcal{F}(X))$ if

1. $\nu(E) \in \text{Eff}(\mathcal{H})$ for every $E \in \mathcal{F}(X)$,
2. $\nu(X) \neq 0$, and
3. for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$ we have
$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)$$
where the convergence on the right side of the previous equality is with respect to the σ-weak topology of $\mathcal{B}(\mathcal{H})$.

We will write $\mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$ for the set of all positive operator valued measures on $(X, \mathcal{F}(X))$ with values in $\mathcal{B}(\mathcal{H})$. With the definition of a positive operator valued measure, we are able to define a positive operator value probability measure, which in turn will be used in the definition of quantum expectation.

Definition 2.2.2. Suppose $\nu \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$ satisfies $\nu(X) = 1 \in \mathcal{B}(H)$, then we say that ν is a positive operator valued probability measure on $(X, \mathcal{F}(X))$, and we write $\mathrm{POVM}^1_{\mathcal{H}}(X, \mathcal{F}(X))$ for the set of all positive operator valued probability measures on $(X, \mathcal{F}(X))$ with values in $\mathcal{B}(\mathcal{H})$. We will sometimes refer to ν as a quantum probability measure.

In the case that $\mathcal{F}(X) = \mathcal{O}(X)$, we will drop the $\mathcal{F}(X)$ from the notation.

Definition 2.2.3. The set of positive operator valued measures on $(X, \mathcal{O}(X))$ is denoted by

$$\mathrm{POVM}_{\mathcal{H}}(X) = \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{O}(X)),$$

and the set of positive operator valued probability measures on $(X, \mathcal{O}(X))$ is denoted by

$$\mathrm{POVM}^1_{\mathcal{H}}(X) = \mathrm{POVM}^1_{\mathcal{H}}(X, \mathcal{O}(X)).$$

If $\nu \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$, then ν induces a finite Borel measure $\mu = \mu_\nu$ on $(X, \mathcal{F}(X))$ given by

$$\mu(E) = \frac{\mathrm{Tr}(\nu(E))}{d} \tag{2.2}$$

for $E \in \mathcal{F}(X)$ where $\mathrm{Tr} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is the trace functional as in Equation 2.1. Note that by definition $\nu(E)$ is a quantum effect so that it is necessarily a positive operator implying that $\mathrm{Tr}(\nu(E))$ is a non-negative real number no greater than d. In particular, since $\nu(X) \neq 0$, it follows that $0 < \mu(X) \leq 1$. However, if $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X, \mathcal{F}(X))$, then the induced finite Borel measure

$$\mu = \mu_\nu = \frac{1}{d} \mathrm{Tr} \circ \nu$$

is, in fact, a Borel probability measure on $(X, \mathcal{F}(X))$.

Suppose $\mathcal{M}(X, \mathcal{F}(X))$ denotes the space of finite Borel measures on $(X, \mathcal{F}(X))$ satisfying $\mu(X) \in (0, 1]$, namely those countably additive set functions $\mu : \mathcal{F}(X) \to [0, 1]$ with $0 < \mu(X) \leq 1$. Then we can identify $\mathcal{M}(X, \mathcal{F}(X))$ with the subset

$$\{\mu \cdot 1 : \mu \in \mathcal{M}(X, \mathcal{F}(X))\} \subseteq \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X)).$$

In particular,

$$\{\mu \cdot 1 | \mu \in \mathcal{M}(X, \mathcal{F}(X)),\ \mu(X) = 1\} \subseteq \mathrm{POVM}^1_{\mathcal{H}}(X, \mathcal{F}(X))$$

so that we can consider ordinary probability measures as scalar-valued positive operator valued probability measures.

That is if $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$, the triple $(X, \mathcal{O}(X), \nu)$ is a quantum probability space while $(X, \mathcal{O}(X), \mu)$ is a classical probability space.

Definition 2.2.4. A function $\psi : X \to \mathcal{B}(\mathcal{H})$ is said to be $\mathcal{F}(X)$-measurable if for every state $\rho \in \mathrm{S}(\mathcal{H})$, the complex-valued function $f = f_\rho : X \to \mathbb{C}$ given by

$$x \mapsto \mathrm{Tr}(\rho\psi(x))$$

is $\mathcal{F}(X)$-measurable in that $f^{-1}(E) \in \mathcal{F}(X)$ for every Borel set $E \in \mathcal{O}(\mathbb{C})$.

In conjunction with the classical definition, we say that a $\mathcal{O}(X)$-measurable function is a quantum random variable.

Definition 2.2.5. A function $\psi : X \to \mathcal{B}(\mathcal{H})$ is called a quantum random variable if the complex-valued functions

$$x \mapsto \mathrm{Tr}(\rho\psi(x))$$

are complex-valued random variables.

We now collect some important results regarding quantum random variables. First, note that the sum and product of quantum random variables is again a quantum random variable. This follows since every quantum random variable $\psi : X \to \mathcal{B}(\mathcal{H})$ induces d^2 complex-valued random variables, and it is a standard exercise in basic probability to show that sums and products of complex-valued random variables are themselves random variables. Second, note that if $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a positive quantum random variable so that $\psi(x) \in B(\mathcal{H})_+$ for every $x \in X$, then $\psi^{1/2}$ is a positive random variable. This fact is Proposition 3.1 of [3].

Suppose that $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable. If $i, j \in \{1, \dots, d\}$, then the function $\psi^{ij} : X \to \mathbb{C}$ given by

$$\psi^{ij}(x) = \langle \psi(x) e_j, e_i \rangle \tag{2.3}$$

is a complex-valued random variable. Conversely, if the d^2 functions

$$\{\psi^{ij} : i, j \in \{1, \dots, d\}\}$$

given by (2.3) are all complex-valued random variables, then ψ is a quantum random variable.

In addition, if $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and $\rho_\xi = \xi \otimes \xi$ so that $\rho_\xi \eta = \langle \eta, \xi \rangle \xi$ for all $\eta \in \mathcal{H}$, then $\mathrm{Tr}(\rho_\xi) = \|\xi\|^2 = 1$ and so

$$\mathrm{Tr}(\rho_\xi \psi(x)) = \langle \psi(x)\xi, \xi \rangle$$

for all $\xi \in \mathcal{H}$ implying that $x \mapsto \mathrm{Tr}(\rho_\xi \psi(x))$ is measurable. The polarisation identity then implies that $x \mapsto \langle \psi(x)\xi, \eta \rangle$ is measurable for all ξ, $\eta \in \mathcal{H}$. Since the set of extreme points of $\mathcal{B}(\mathcal{H})$, namely $\{\rho_\xi : \xi \in \mathcal{H}, \|\xi\| = 1\}$, is convex, it follows that

$$x \mapsto \mathrm{Tr}(\rho \psi(x))$$

is measurable for every state $\rho \in \mathrm{S}(\mathcal{H})$. Hence, to show that the function $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable, it suffices to show that

$$x \mapsto \langle \psi(x)\xi, \eta \rangle \tag{2.4}$$

is a complex-valued random variable for every ξ, $\eta \in \mathcal{H}$.

Proposition 2.2.6. *If $\psi_1 : X \to \mathcal{B}(\mathcal{H})$ and $\psi_2 : X \to \mathcal{B}(\mathcal{H})$ are quantum random variables, then so are $\psi_1 + \psi_2$ and $\psi_1\psi_2$.*

Proof. Since the trace functional is additive we can immediately conclude that $\psi_1+\psi_2$ is a quantum random variable. That is, since both $x \mapsto \mathrm{Tr}(\rho\psi_1(x))$ and $x \mapsto \mathrm{Tr}(\rho\psi_2(x))$ are complex-valued random variables for every state $\rho \in \mathrm{S}(\mathcal{H})$, we conclude that

$$x \mapsto \mathrm{Tr}(\rho(\psi_1(x) + \psi_2(x))) = \mathrm{Tr}(\rho\psi_1(x) + \rho\psi_2(x)) = \mathrm{Tr}(\rho\psi_1(x)) + \mathrm{Tr}(\rho\psi_2(x))$$

is a complex-valued random variable for every state $\rho \in \mathrm{S}(\mathcal{H})$. In order to show that $\psi_1\psi_2$ is a quantum random variable, we will make use of the observation (2.4). That is, by writing

$$\langle \psi_1(x)\psi_2(x)e_j, e_i\rangle = \sum_{k=1}^{d} \langle \psi_1(x)e_k, e_i\rangle\langle \psi_2(x)e_j, e_k\rangle$$

so that $x \mapsto \langle \psi_1(x)\psi_2(x)e_j, e_i\rangle$ can be expressed as the sum and product of complex-valued random variables, we conclude that

$$x \mapsto \langle \psi_1(x)\psi_2(x)\xi, \eta\rangle$$

is a complex-valued random variable for every $\xi, \eta \in \mathcal{H}$. $\square$

For a proof of the following result, see Proposition 2.1 in [3].

Proposition 2.2.7. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a positive quantum random variable, then $\psi^{1/2}$ is a positive quantum random variable.*

Note that as a consequence of the previous two results, it follows that if $\psi_1 : X \to \mathcal{B}(\mathcal{H})_+$ and $\psi_2 : X \to \mathcal{B}(\mathcal{H})_+$ are both positive quantum random variables, then

$$\psi_2^{1/2}\psi_1\psi_2^{1/2} \quad \text{and} \quad \psi_1^{1/2}\psi_2\psi_1^{1/2} \tag{2.5}$$

are both positive quantum random variables.

2.3 The Principal Radon-Nikodým Derivative

In this section, we recall the definition of the principal Radon-Nikodým derivative, which is introduced in [3]. It will be used in section 2.4 to define quantum expectation.

Suppose that $\nu_1,\ \nu_2 \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$. Then we say that ν_2 is absolutely continuous with respect to ν_1, written $\nu_2 \ll_{\mathrm{ac}} \nu_1$, if $\nu_2(E) = 0$ for every $E \in \mathcal{F}(X)$ with $\nu_1(E) = 0$. For a given positive operator valued measure ν, we wish to compare it to its induced measure $\mu = \mu_\nu$. We will do so by identifying μ with $\mu \cdot 1 \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$. Then since the trace functional maps non-zero positive operators to strictly positive real numbers, we see that $\nu \ll_{\mathrm{ac}} \mu$ and $\mu \ll_{\mathrm{ac}} \nu$ so that μ and ν are mutually absolutely continuous.

Definition 2.3.1. Suppose that $\nu \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$ and that $\mu = \mu_\nu$ is the induced finite Borel measure as given in (2.2). If $\nu_{ij} : \mathcal{F}(X) \to \mathbb{C}$ is the measure defined by $\nu_{ij}(E) = \langle \nu(E)e_j, e_i \rangle$, then $\nu_{ij} \ll_{\mathrm{ac}} \mu$ so by the classical Radon-Nikodým theorem, there exists a μ-almost everywhere unique function

$$\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu} \in L^1(X, \mathcal{F}(X), \mu)$$

such that

$$\nu_{ij}(E) = \int_E \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu}\, \mathrm{d}\mu$$

for all $E \in \mathcal{F}(X)$. The function

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \mathcal{B}(\mathcal{H})$$

given by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \sum_{i,j=1}^{d} \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu} \otimes e_{ij}$$

is called the principal Radon-Nikodým derivative of ν.

As is shown in [3],

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \mathcal{B}(\mathcal{H})$$

is $\mathcal{F}(X)$-measurable and hence a quantum random variable. Moreover, it is also shown that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)$$

is a positive operator for μ-almost all $x \in X$ so that for such x it is possible to consider

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}.$$

Therefore, we can define a positive quantum random variable

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} : X \to \mathcal{B}(\mathcal{H})_+$$

by setting it equal to

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}$$

at those $x \in X$ for which

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)$$

is a positive operator and equal to 0 otherwise. That is, if $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a positive quantum random variable, then

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} \geq 0 \tag{2.6}$$

as mentioned in (2.5). Finally, it is worth noting that the principal Radon-Nikodým derivative of ν does depend on the choice of orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathcal{H}$. However, if $\{e_1', \ldots, e_n'\}$ is another orthonormal basis of $\mathcal{H}$, then the principal Radon-Nikodým derivative of ν in this new basis is just the principal Radon-Nikodým derivative of $\theta \circ \nu$ in the original basis where θ is the automorphism induced by the unitary operator that changes the new basis to the original basis.

2.4 Quantum Expectation

In this section we recall the definition of a new type of operator valued integral first introduced in [3], which integrates quantum random variables with respect to positive operator valued measures.

Definition 2.4.1. Suppose that $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ is a positive operator valued measure and that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \mathcal{B}(\mathcal{H})$$

is the principal Radon-Nikodým derivative of ν. If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable, then ψ is said to be ν-integrable if for every state $\rho \in \mathrm{S}(\mathcal{H})$ the complex-valued function $\psi_\rho : X \to \mathbb{C}$ given by

$$\psi_\rho(x) = \mathrm{Tr}\left(\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right) \tag{2.7}$$

is μ-integrable.

Definition 2.4.2. Suppose $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$. If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν-integrable quantum random variable, then the integral of ψ with respect to ν is the unique operator $I(\psi;\nu) : \mathcal{H} \to \mathcal{H}$ such that

$$\mathrm{Tr}\left(\rho I(\psi;\nu)\right) = \int_X \psi_\rho \, \mathrm{d}\mu$$

for every state $\rho \in \mathrm{S}(\mathcal{H})$ where ψ_ρ is given by (2.7). We then write the integral of ψ with respect to ν as

$$I(\psi;\nu) = \int_X \psi \, \mathrm{d}\nu.$$

If $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ is a positive operator valued probability measure, then the induced measure $\mu = \mu_\nu$ is a Borel probability measure, and so it is appropriate to write

$$\mathbb{E}_\mu\left[\psi_\rho\right] = \int_X \psi_\rho \,\mathrm{d}\mu.$$

Therefore, we will write $\mathbb{E}_\nu\left[\psi\right] = I(\psi;\nu)$ for the unique operator acting on $\mathcal{H}$ having the property that

$$\mathrm{Tr}\left(\rho\mathbb{E}_\nu\left[\psi\right]\right) = \mathbb{E}_\mu\left[\psi_\rho\right]$$

for every state $\rho \in \mathrm{S}(\mathcal{H})$ and say that $\mathbb{E}_\nu\left[\psi\right]$ is the quantum expectation (or quantum expected value) of the ν-integrable quantum random variable ψ with respect to the positive operator valued probability measure ν. Note that we will sometimes write

$$\mathbb{E}_\nu\left[\psi\right] = \int_X \psi \,\mathrm{d}\nu$$

for the quantum expectation. We will also need to consider integrating over sets other than just X itself. Therefore, if $E \in \mathcal{O}(X)$, let

$$\int_E \psi \,\mathrm{d}\nu = \int_X \psi\chi_E \,\mathrm{d}\nu = \mathbb{E}_\nu\left[\psi\chi_E\right]$$

where χ_E denotes the characteristic (or indicator) function of the measurement outcome $E \in \mathcal{O}(X)$.

We will now list several important properties of quantum expectation whose proofs can be found in either [2] or [3].

Proposition 2.4.3. *Let $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$. The quantum expectation of an effect-valued ν-integrable quantum random variable is an effect.*

Corollary 2.4.4. *Quantum expectation is monotonic in the sense that if ν is a positive operator valued probability measure and $\psi :\to \mathcal{B}(\mathcal{H})_+$ is a positive ν-integrable quantum random variable, then $\mathbb{E}_\nu [\psi]$ is a positive operator.*

Proposition 2.4.5. *Quantum expectation is a linear operator in the sense that if $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$, $\psi_1 : X \to \mathcal{B}(\mathcal{H})$ and $\psi_2 : X \to \mathcal{B}(\mathcal{H})$ are ν-integrable quantum random variables, and ϱ_1, $\varrho_2 \in \mathcal{B}(\mathcal{H})$ commute with the range of $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then*

$$\mathbb{E}_\nu [\varrho_1\psi_1 + \varrho_2\psi_2] = \varrho_1\mathbb{E}_\nu [\psi_1] + \varrho_2\mathbb{E}_\nu [\psi_2] .$$

Proposition 2.4.6. *Quantum expectation is additive in the sense that if ν is a positive operator valued probability measure and $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν-integrable quantum random variable, and $E, F \in \mathcal{O}(X)$ with $E \cap F = \emptyset$, then*

$$\int_{E\cup F} \psi \, \mathrm{d}\nu = \int_E \psi \, \mathrm{d}\nu + \int_F \psi \, \mathrm{d}\nu.$$

The following result indicates the structure of finitely supported $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and their Radon-Nikodým derivatives. This result will be used to calculate counter examples, which are proved using a finite dimensional sample space.

Proposition 2.4.7. *Suppose that X is a finite set, say $X = \{x_1, \ldots, x_n\}$, and let $\mathcal{O}(X) = 2^X$ be the power set of X. Suppose further that $h_1, \ldots, h_n \in \mathcal{B}(\mathcal{H})_+$ are non-zero and satisfy $h_1 + \cdots + h_n = 1$. If $\psi : X \to \mathcal{B}(\mathcal{H})$ is any function and if $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ is defined by*

$$\nu(E) = \sum_{j=1}^{n} \delta_{x_j}(E) h_j = \sum_{x_j \in E} h_j$$

for any $E \in \mathcal{O}(X)$, then ψ is ν-integrable and

$$\mathbb{E}_\nu [\psi] = \sum_{j=1}^{n} h_j^{1/2} \psi(x_j) h_j^{1/2}.$$

As we have shown, quantum expectation possesses many properties of classical expectation, namely quantum expectation is monotone, additive, and linear. Moreover, when the underlying space X is finite, the quantum expectation is given by a sum which is analogous to the situation classically for the expected value of a discrete random variable. Furthermore, if ψ is a quantum random variable and if ψ^* is defined by $\psi^*(x) = \psi(x)^*$, $x \in X$, then it is easily verified that

$$\mathbb{E}_\nu [\psi^*] = \mathbb{E}_\nu [\psi]^* . \tag{2.8}$$

2.5 Quantum Conditional Expectation

In this section we recall the notion of quantum conditional expectation first studied in [2] and collect important results relating to this notion. As in the classical setting, the existence of quantum conditional expectation is proved using a quantum version of the Radon-Nikodým derivative, namely the non-principal Radon-Nikodým derivative introduced in [2].

First, we begin with the definition and existence of quantum conditional expectation as proved in [2].

Theorem 2.5.1. *Suppose that $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and that $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$. If $\mathcal{F}(X)$ is a sub-σ-algebra of $\mathcal{O}(X)$, then there exists a function $\varphi : X \to \mathcal{B}(\mathcal{H})$ such that*

1. *φ is $\mathcal{F}(X)$-measurable,*
2. *φ is ν-integrable, and*
3. *$\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\varphi\chi_E]$ for every $E \in \mathcal{F}(X)$.*

Moreover, if $\tilde{\varphi}$ is any other ν-integrable $\mathcal{F}(X)$-measurable function satisfying

$$\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\tilde{\varphi}\chi_E]$$

for every $E \in \mathcal{F}(X)$, then $\nu(\{x \in X : \varphi(x) \neq \tilde{\varphi}(x)\}) = 0$.

Definition 2.5.2. Suppose that $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and that $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$. Suppose further that $\mathcal{F}(X)$ is a sub-σ-algebra of $\mathcal{O}(X)$. A quantum random variable $\varphi : X \to \mathcal{B}(\mathcal{H})$ satisfying the three properties of Theorem 2.5.1 is called a version of quantum conditional expectation of ψ given $\mathcal{F}(X)$ relative to ν and is denoted by $\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$.

A consequence of Theorem 2.5.1 is that any two versions φ and $\tilde{\varphi}$ of $\mathbb{E}_\nu[\psi|\mathcal{F}(X)]$ satisfy $\nu(\{x \in X : \varphi(x) \neq \tilde{\varphi}(x)\}) = 0$. Thus, instead of saying "$\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$ ν-almost surely" we will identify different versions and say that $\mathbb{E}_\nu[\psi|\mathcal{F}(X)]$ is *the* quantum conditional expectation of ψ given $\mathcal{F}(X)$ relative to ν. Hence, $\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$ is an $\mathcal{F}(X)$-measurable quantum random variable $\varphi : X \to \mathcal{B}(\mathcal{H})$ with the property that

$$\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\varphi\chi_E]$$

for every $E \in \mathcal{F}(X)$.

The following propositions are found in [2]. However, we include the proofs for completeness because they illustrate how to work with the quantum conditional expectation.

Proposition 2.5.3. *If $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$, then*

$$\mathbb{E}_\nu[\psi] = \mathbb{E}_\nu[\mathbb{E}_\nu[\psi|\mathcal{F}(X)]].$$

Proof. The quantum conditional expectation $\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$ satisfies

$$\int_E \psi \, \mathrm{d}\nu = \int_E \varphi \, \mathrm{d}\nu$$

for every $E \in \mathcal{F}(X)$. Since $X \in \mathcal{F}(X)$, we conclude

$$\mathbb{E}_\nu[\psi] = \int_X \psi \, \mathrm{d}\nu = \int_X \varphi \, \mathrm{d}\nu = \int_X \mathbb{E}_\nu[\psi|\mathcal{F}(X)] \, \mathrm{d}\nu = \mathbb{E}_\nu[\mathbb{E}_\nu[\psi|\mathcal{F}(X)]]$$

as required. □

Proposition 2.5.4. *If $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable $\mathcal{F}(X)$-measurable quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$, then $\mathbb{E}_\nu[\psi|\mathcal{F}(X)] = \psi$.*

Proof. Suppose that $\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$. Since $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is assumed to be $\mathcal{F}(X)$-measurable and ν-integrable, and since it is a tautology that $\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\psi\chi_E]$ for every $E \in \mathcal{F}(X)$, we conclude from Theorem 2.5.1 that $\nu(\{x \in X : \varphi(x) \neq \psi(x)\}) = 0$. Hence, $\mathbb{E}_\nu[\psi|\mathcal{F}(X)] = \psi$ (ν-almost surely) as required. □

Proposition 2.5.5. *Let $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and let $\psi_1 : X \to \mathcal{B}(\mathcal{H})_+$ and $\psi_2 : X \to \mathcal{B}(\mathcal{H})_+$ be ν-integrable $\mathcal{F}(X)$-measurable quantum random variables with $\mathbb{E}_\nu[\psi_1] \neq 0$ and $\mathbb{E}_\nu[\psi_2] \neq 0$. If $\varrho_1, \varrho_2 \in \mathcal{B}(\mathcal{H})$ commute with the range of $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then*

$$\mathbb{E}_\nu[\varrho_1\psi_1 + \varrho_2\psi_2|\mathcal{F}(X)] = \varrho_1\mathbb{E}_\nu[\psi_1|\mathcal{F}(X)] + \varrho_2\mathbb{E}_\nu[\psi_2|\mathcal{F}(X)].$$

Proof. Suppose that $\varphi_1 = \mathbb{E}_\nu\left[\psi_1 | \mathcal{F}(X)\right]$ and $\varphi_2 = \mathbb{E}_\nu\left[\psi_2 | \mathcal{F}(X)\right]$ so that

$$\int_E \varphi_1 \, \mathrm{d}\nu = \int_E \psi_1 \, \mathrm{d}\nu \quad \text{and} \quad \int_E \varphi_2 \, \mathrm{d}\nu = \int_E \psi_2 \, \mathrm{d}\nu$$

for every $E \in \mathcal{F}(X)$. Suppose further that $\varphi = \mathbb{E}_\nu\left[\varrho_1\psi_1 + \varrho_2\psi_2 | \mathcal{F}(X)\right]$ so that

$$\int_E \varphi \, \mathrm{d}\nu = \int_E (\varrho_1\psi_1 + \varrho_2\psi_2) \, \mathrm{d}\nu$$

for every $E \in \mathcal{F}(X)$. Since quantum expectation is a linear operator (see Proposition 2.4.5), we conclude that

$$\begin{aligned}\int_E (\varrho_1\psi_1 + \varrho_2\psi_2) \, \mathrm{d}\nu = \int_X (\varrho_1\psi_1 + \varrho_2\psi_2)\chi_E \, \mathrm{d}\nu &= \varrho_1 \int_X \psi_1\chi_E \, \mathrm{d}\nu + \varrho_2 \int_X \psi_2\chi_E \, \mathrm{d}\nu \\ &= \varrho_1 \int_E \psi_1 \, \mathrm{d}\nu + \varrho_2 \int_E \psi_2 \, \mathrm{d}\nu\end{aligned}$$

for every $E \in \mathcal{F}(X)$ which implies that

$$\int_E \varphi \, \mathrm{d}\nu = \varrho_1 \int_E \varphi_1 \, \mathrm{d}\nu + \varrho_2 \int_E \varphi_2 \, \mathrm{d}\nu$$

for every $E \in \mathcal{F}(X)$ and so $\mathbb{E}_\nu\left[\varrho_1\psi_1 + \varrho_2\psi_2 | \mathcal{F}(X)\right] = \varrho_1\mathbb{E}_\nu\left[\psi_1 | \mathcal{F}(X)\right] + \varrho_2\mathbb{E}_\nu\left[\psi_2 | \mathcal{F}(X)\right]$ (ν-almost surely) as required. □

2.6 A Non-Commutative Multiplication

This section reviews a non-commutative multiplication introduced in [2] and the non-principal Radon-Nikodým derivative introduced in [3], which is a generalization of the principal Radon-Nikodým derivative. It allows one to relate two absolutely continuous positive operator valued probability measures.

The following theorem proving existence of the non-principal Radon-Nikodým derivative is from [3].

Theorem 2.6.1. *If* $\nu_1, \nu_2 \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$ *then the following are equivalent*

1. $\nu_2 \ll_{\mathrm{ac}} \nu_1$,

2. *There exists* $\varphi : (X, \mathcal{F}(X)) \to \mathcal{B}(\mathcal{H})$, *a bounded* ν_1*-integrable* $\mathcal{F}(X)$*-measurable* ν_1*-almost everywhere unique function, such that for every* $E \in \mathcal{F}(X)$

$$\nu_2(E) = \int_E \varphi \, \mathrm{d}\nu_1.$$

If the above conditions hold, and if $\mu_j = \mu_{\nu_j}$ *is the induced measure of* ν_j*, then* $\mu_2 \ll_{\mathrm{ac}} \mu_1$ *and*

$$\varphi = \left(\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}\right) \left[\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1/2} \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2} \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1/2} \right].$$

In accordance with the classical Radon-Nikodým derivative, denote the non-principal Radon-Nidodým derivative by

$$\varphi = \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}. \tag{2.9}$$

It should be noted that the non-principal Radon-Nikodým derivative is a positive operator valued random variable, which is apparent from Equation 2.9.

In order to introduce the non-commutative multiplication, one first needs to recall the geometric mean of two positive operators. See [16] for details.

Definition 2.6.2. Suppose $a, b \in \mathcal{B}(\mathcal{H})_+$ are both invertible, then the geometric mean of a and b is

$$a\#b = a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{1/2}a^{1/2}.$$

If $a, b \in \mathcal{B}(\mathcal{H})_+$ are not invertible, then

$$a\#b = \lim_{\varepsilon \to 0^+}(a + \varepsilon 1)\#(b + \varepsilon 1)$$

where convergence is with respect to the strong operator topology

With the definition of the geometric mean, one is able to define the non-commutative multiplication.

Definition 2.6.3. Suppose that $\nu_1, \nu_2 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_2 \ll_{\mathrm{ac}} \nu_1$. For $j = 1, 2$, let $\mu_j = \nu_{v_j}$ denote the induced measure of ν_j. Then given a quantum random variable $\psi : X \to \mathcal{B}(\mathcal{H})$, define

$$\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \left(\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1} \# \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\right)\left(\frac{\mathrm{d}\nu_1}{d\mu_1}\right)^{1/2}\psi\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2}\left(\left(\frac{\mathrm{d}\nu_1}{d\mu_1}\right)^{-1} \# \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\right). \quad (2.10)$$

If ν_1 is taken to be μ_2 in the previous definition, the formula (2.10) simplifies.

Theorem 2.6.4. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable and ν is a positive operator valued probability measure, then*

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}\psi\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}.$$

Proof. Suppose $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable and $\nu_1 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with its associated Borel measure $\mu_1 = \mu_{\nu_1}$. Then viewing μ as a positive operator valued measure, denoted $\nu_2 = \mu_1 \cdot 1$, it follows that $\nu_1 \ll_{\mathrm{ac}} \nu_2$. Notice that the induced measure of ν_2 is the real valued set function $\mu_2 = \mu_{\nu_2} : \mathcal{O}(X) \to \mathbb{R}$ such that

$$\mu_2(E) = \frac{\mathrm{Tr}(\nu_2(E))}{d} = \frac{\mathrm{Tr}(\mu_1(E)\cdot 1)}{d} = \mu_1(E)\frac{\mathrm{Tr}(1)}{d} = \mu_1(E)\ \frac{d}{d} = \mu_1(E)$$

so that $\mu_2 = \mu_1$. Then the non-principal Radon-Nikodým derivative of ν_1 with respect to ν_2 is,

$$\begin{aligned}
\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} &= \left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_2}\right)\left[\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}\right] \\
&= \left(\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_2}\right)\left[\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}\right] \\
&= \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}.
\end{aligned}$$

Then to see that

$$\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} = 1$$

notice that

$$\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} = \sum_{i,j=1}^{d} \frac{\mathrm{d}\nu_{2_{ij}}}{\mathrm{d}\mu} \otimes e_{ij}$$

where $\nu_{2_{ij}}$ is the set function $\nu_{2_{ij}} : \mathcal{O}(X) \to \mathbb{C}$ such that $\nu_{2_{ij}}(E) = \langle \nu_2(E)e_j, e_i\rangle$. That is

$$\nu_{2_{ij}}(E) = \langle \nu_2(E)e_j, e_i\rangle = \langle \mu_2(E)\cdot 1e_j, e_i\rangle = \mu_2(E)\langle 1e_j, e_i\rangle = \mu_2(E)\delta_{ij}$$

where δ_{ij} is the Kronecker delta function. Then

$$\frac{\mathrm{d}\nu_{2_{ij}}}{\mathrm{d}\mu} \in L^1(X, \mathcal{O}(X), \mu)$$

is the unique function such that

$$\nu_{2_{ij}}(E) = \int_E \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu}\,\mathrm{d}\mu.$$

Then if $i = j$, it follows that $\nu_{2_{ii}}(E) = \mu(E)$ so that

$$\mu(E) = \int_E \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu}\,\mathrm{d}\mu,$$

which implies that

$$\frac{\mathrm{d}\nu_{ii}}{\mathrm{d}\mu} = 1.$$

In the case that $i \neq j$, it follows that $\nu_{2_{ij}}(E) = 0$ so that

$$0 = \int_E \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu}\,\mathrm{d}\mu$$

for all $E \in \mathcal{O}(X)$, which implies that

$$\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu} = 0.$$

Then

$$\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} = \sum_{i,j=1}^{d} \frac{\mathrm{d}\nu_{2_{ij}}}{\mathrm{d}\mu} \otimes e_{ij} = \sum_{i,j=1}^{d} \delta_{ij} \otimes e_{ij} = I_d$$

where I_d is the $d \times d$ identity matrix, and so

$$\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} = 1.$$

Then it follows that

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}.$$

Now,

$$\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1} \# \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} = \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2} \left[\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2} \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2}\right] \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1/2}$$
$$= \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}$$
$$= \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}.$$

Then it follows that

$$\psi \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} = \left(\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1} \# \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}\right) \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2} \left(\left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{-1} \# \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}\right)$$
$$= \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right)^{1/2}$$
$$= \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2}.$$

Recall that ν_2 is the positive operator valued measure such that $\nu_2 = \mu_1 \cdot 1$. That is

$$\psi \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} = \psi \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}(\mu_1 \cdot 1)} = \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2}.$$

In order to simplify notation, μ_1 will be assumed to be either a positive operator valued measure, or an induced measure as needed so that

$$\psi \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1} = \psi \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}(\mu_1 \cdot 1)} = \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2},$$

as required. $\square$

Chapter 3

Quantum Random Variables

The purpose of this chapter is to establish the first of our primary results of the thesis. Included are limiting results for quantum random variables and a discussion about quantum random variables with mean zero. The section on limiting results will detail quantum analogues of classical results such as the the continuity of quantum expectation, which is the quantum analogue of Lebesgue's dominated convergence theorem. The results in this section will prove vital in Chapter 4. The section on quantum random variables with mean zero provides a partial classification of mean zero quantum random variables. This classification is used in Chapter 4. Again we stress that although the quantum statements mimic their classical counterparts, the techniques of proof are decidedly non-classical.

3.1 Continuity of Quantum Expectation

In this section we will establish limiting results for sequences of quantum random variables, namely a result concerning the continuity of quantum expectation of a quantum random variable in Theorem 3.1.4. It is the natural quantum analogue of the classical dominated convergence theorem and we will use it in the Chapter 4.

Definition 3.1.1. Suppose $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of quantum random variables. We say the sequence $\{\psi_n\}_{n=1}^{\infty}$ converges ultraweakly μ-almost surely to the quantum random variable ψ if $\mathrm{Tr}(\rho\psi_n(x)) \to \mathrm{Tr}(\rho\psi(x))$ for all $\rho \in S(\mathcal{H})$ and μ-almost all $x \in X$.

Lemma 3.1.2. *Suppose $\psi_n : X \to \mathcal{B}(\mathcal{H})$ is a sequence of quantum random variables. If $\psi_n \to \psi$ ultraweakly μ-almost surely, then $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable.*

Proof. Since $\psi_n \to \psi$ ultraweakly μ-almost surely, it follows that $\mathrm{Tr}(\rho\psi_n(x)) \to \mathrm{Tr}(\rho\psi(x))$ for all $\rho \in S(\mathcal{H})$ and μ-almost all $x \in X$. But since each $\mathrm{Tr}(\rho\psi_n(x))$ is a complex valued random variable, the limit of the sequence $\{\mathrm{Tr}(\rho\psi_n(x))\}_{n\in\mathbb{N}}$ converges to a complex valued random variable, namely $\mathrm{Tr}(\rho\psi(x))$ for each $x \in X$, and therefore ψ is a quantum random variable. $\square$

Lemma 3.1.3. *Suppose a sequence of quantum random variables $\{\psi_n\}_{n=1}^{\infty}$ converges ultraweakly μ-almost surely to ψ. Then $\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ converges ultraweakly μ-almost surely to $\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.*

Proof. Notice that for any $\rho \in S(\mathcal{H})$,

$$\operatorname{Tr}\left(\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi_n(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right) = \operatorname{Tr}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi_n(x)\right).$$

Then define

$$\tilde{\rho}_x = \left[\operatorname{Tr}\left(\rho\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)\right]^{-1}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right)$$

and notice that $\tilde{\rho}_x \in S(\mathcal{H})$. But by assumption ψ_n converges ultraweakly μ-almost surely to ψ. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\operatorname{Tr}\left(\rho\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x)\right) &= \lim_{n\to\infty}\operatorname{Tr}\left(\rho\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)\operatorname{Tr}(\tilde{\rho}_x\psi_n(x)) \\
&= \operatorname{Tr}\left(\rho\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)\operatorname{Tr}(\tilde{\rho}_x\psi(x)) \\
&= \operatorname{Tr}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\right) \\
&= \operatorname{Tr}\left(\rho\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x)\right)
\end{aligned}$$

as required. □

Now we prove the quantum analogue of Lebesgue's dominated convergence theorem, namely the continuity of quantum expectation. This is an important result that will be used extensively throughout the rest of the thesis.

Theorem 3.1.4 (Continuity of Quantum Expectation)**.** *Let ψ_n be a sequence of ν-integrable quantum random variables. If ψ_n converges ultraweakly μ-almost surely to ψ, and if there exists a μ-integrable random variables $Z_\rho : X \to \mathbb{C}$ such that $\left|\psi_\rho^{(n)}\right| \leq Z_\rho$ almost surely for all $\rho \in S(\mathcal{H})$, then ψ is ν-integrable and $\mathbb{E}_\nu [\psi_n] \to \mathbb{E}_\nu [\psi]$*

Proof. By the previous lemma,

$$\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \to \psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$$

ultraweakly μ-almost surely. Now define the sequence of complex valued random variables $\{\zeta_\rho^{(n)}\}_{n\in\mathbb{N}}$ such that

$$\zeta_\rho^{(n)} = \mathrm{Tr}\left(\rho\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right).$$

Notice that

$$\begin{aligned}
\lim_{n\to\infty} \zeta_\rho^{(n)} &= \lim_{n\to\infty} \mathrm{Tr}\left(\rho\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)\\
&= \mathrm{Tr}\left(\lim_{n\to\infty} \rho\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)\\
&= \mathrm{Tr}\left(\rho \lim_{n\to\infty}\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)\\
&= \mathrm{Tr}\left(\rho\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)\\
&= \zeta_\rho
\end{aligned}$$

where the second equality follows from the continuity of the trace operator, and the last equality follows from Lemma 3.1.3 so that $\zeta_\rho^{(n)}$ converges pointwise μ-almost everywhere to ζ_ρ. Then notice that the sequence $\left\{\zeta_\rho^{(n)}\right\}$ is bounded by a μ-integrable random variable $Z_\rho : X \to \mathbb{C}$ by assumption. Thus by Lebesgue's dominated convergence theorem,

$$\mathrm{Tr}\left(\rho\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)$$

is a μ-integrable random variable and for every $\rho \in S(\mathcal{H})$

$$\int_X \mathrm{Tr}\left(\rho\left(\psi_n \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right) \mathrm{d}\mu \to \int_X \mathrm{Tr}\left(\rho\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right) \mathrm{d}\mu.$$

Therefore ψ is a ν-integrable function and $\mathrm{Tr}(\rho\mathbb{E}_\nu[\psi_n]) \to \mathrm{Tr}(\rho\mathbb{E}_\nu[\psi])$ which implies that $\mathbb{E}_\nu[\psi_n] \to \mathbb{E}_\nu[\psi]$ ultraweakly. $\square$

As a first application of the continuity of quantum expectation we prove that, under certain conditions, quantum expectation is linear over infinite sums. In fact, this could even be considered as a special case of a quantum Fubini-type theorem.

Theorem 3.1.5. *Suppose that ψ_n, $n \in \mathbb{N}$ is a sequence of ν-integrable quantum random variables. Suppose further that*

$$\sum_{n=1}^{\infty} \psi_n = \lim_{N\to\infty} \sum_{n=1}^{N} \psi_n$$

exists where convergence is with respect to the σ-weak topology of $\mathcal{B}(\mathcal{H})$. Then

$$\sum_{n=1}^{\infty} \psi_n$$

is a ν-integrable quantum random variable with

$$\mathbb{E}_\nu\left[\sum_{n=1}^{\infty} \psi_n\right] = \sum_{n=1}^{\infty} \mathbb{E}_\nu[\psi_n].$$

Proof. Let $\varphi_N = \sum_{n=1}^{N} \psi_n$ so that $\varphi_N \to \varphi$ ultraweakly μ-almost surely where $\varphi = \sum_{n=1}^{\infty} \psi_n$. By Lemma 3.1.2, it follows that φ is a quantum random variable. Then by Theorem 3.1.4, φ is ν-integrable and

$$\lim_{N\to\infty} \mathbb{E}_\nu\left[\varphi_N\right] = \mathbb{E}_\nu\left[\varphi\right]. \tag{3.1}$$

However,

$$\begin{aligned}\mathbb{E}_\nu\left[\varphi_N\right] &= \mathbb{E}_\nu\left[\sum_{n=1}^{N} \psi_n\right] \\ &= \sum_{j=1}^{N} \mathbb{E}_\nu\left[\psi_n\right]\end{aligned}$$

which follows from finite additivity of quantum expectation, which is proved in [2].

Thus, by Equation (3.1),

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{E}_\nu\left[\psi_n\right] &= \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{E}_\nu\left[\psi_n\right] \\ &= \lim_{N\to\infty} \mathbb{E}_\nu\left[\varphi_N\right] \\ &= \mathbb{E}_\nu\left[\varphi\right] \\ &= \mathbb{E}_\nu\left[\sum_{n=1}^{\infty} \psi_n\right]\end{aligned}$$

as desired. $\square$

The following example makes use of the results above.

Example 3.1.6. If ψ is an effect valued quantum random variable such that $\psi(x) \neq 0$ and $\psi(x) \neq 1$ for all $x \in X$, then

$$\sum_{n=1}^{\infty} \mathbb{E}_{\nu} \left[\psi[1-(1+\psi^{-2})^{-1}]^n \psi\right] = 1.$$

In order to show this, we will use Theorem 3.1.5 to establish that

$$\sum_{n=1}^{\infty} \mathbb{E}_{\nu} \left[\psi[1-(1+\psi^{-2})^{-1}]^n \psi\right]$$

and

$$\mathbb{E}_{\nu} \left[\psi \left(\sum_{n=1}^{\infty} [1-(1+\psi^{-2})^{-1}]^n\right) \psi\right]$$

are equal. We will first show that the series

$$\sum_{n=1}^{\infty} [1-(1+\psi^{-2})^{-1}]^n$$

is well defined.

First, consider the matrix A with $\sigma(A) \in (0,1)$. Then

$$\sum_{n=1}^{\infty} A^n = (1-A)^{-1} - 1.$$

Now, with $A = 1-(1+\psi^{-2})^{-1}$, we have that

$$\sum_{n=1}^{\infty} [1-(1+\psi^{-2})^{-1}]^n = (1+\psi^{-2}) - 1 = \psi^{-2}.$$

Since the series is well defined, Theorem 3.1.5 says that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}_{\nu} \left[\psi[1-(1+\psi^{-2})^{-1}]^n \psi\right] &= \mathbb{E}_{\nu} \left[\psi \left(\sum_{n=1}^{\infty} [1-(1+\psi^{-2})^{-1}]^n\right) \psi\right] \\ &= \mathbb{E}_{\nu} \left[\psi \psi^{-2} \psi\right] \\ &= 1 \end{aligned}$$

as required.

3.2 Quantum Random Variables with Quantum Expectation Zero

The operator valued nature of quantum expectation allows for a greater class of functions to integrate to zero than one would expect from a classical viewpoint. In particular, any function who's range is orthogonal to the principal Radon-Nikodým derivative will have quantum expectation zero. We will show that if ψ is a quantum random variable, then

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} \psi(x) \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = 0$$

for μ-almost all $x \in X$ implies that $\mathbb{E}_\nu[\psi] = 0$. Functions that satisfy these equalities will be used to interpret results in Chapter 4. In this section, the set of all positive operator valued quantum random variables of mean zero will be completely characterized while the set of all mean zero quantum random variables will be partially characterized.

We will begin by providing a rigorous argument that relates a mean zero quantum random variable, ψ, to the statement

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = 0$$

for μ-almost all $x \in X$.

Proposition 3.2.1. *Suppose ψ is a quantum random variable such that*

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = 0$$

for μ-almost all $x \in X$. Then $\mathbb{E}_\nu[\psi] = 0$.

Proof. Assuming the hypotheses, it follows that $\psi_\rho(x) = 0$ for μ-almost all $x \in X$. Then $\mathbb{E}_\nu[\psi]$ is the unique operator such that

$$\mathrm{Tr}(\rho \mathbb{E}_\nu[\psi]) = \int_X \psi_\rho \, \mathrm{d}\mu = 0$$

for all $\rho \in S(\mathcal{H})$. Therefore, since the equality holds for all states $\rho \in S(\mathcal{H})$, it follows that $\mathbb{E}_\nu[\psi] = 0$. $\square$

Now we establish results that relate the range and kernel of an operator to its square root. These results allow for a more natural formulation of theorems.

Lemma 3.2.2. *If $z \in \mathcal{B}(\mathcal{H})_+$, then $\ker(z) = \ker(z^{1/2})$ and $\mathrm{Ran}(z) = \mathrm{Ran}(z^{1/2})$.*

Proof. If $\eta \in \ker(z^{1/2})$, then $z^{1/2}\eta = 0$. This implies that $z\eta = z^{1/2}z^{1/2}\eta = 0$ so that $\eta \in \ker(z)$.

Conversely, suppose that $\eta \in \ker(z)$. This implies that $z\eta = 0$ so that

$$0 = \langle z\eta, \eta \rangle = \langle z^{1/2}\eta, z^{1/2}\eta \rangle.$$

Therefore $z^{1/2}\eta = 0$ so $\eta \in \ker(z^{1/2})$. Then it follows from the orthogonal decomposition $\mathcal{H} = \ker(z^*) \oplus \mathrm{Ran}(z)$ that since $z \in \mathcal{B}(\mathcal{H})_+$, $z = z^*$ and $\mathrm{Ran}(z) = \mathrm{Ran}(z^{1/2})$. $\square$

A useful application of the previous lemma is that

$$\operatorname{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right) = \operatorname{Ran}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right).$$

The result is realized since

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} : X \to \mathcal{B}(\mathcal{H})_{+}.$$

Theorem 3.2.3. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable, then*

$$\psi^{*}\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0 \quad \textit{if and only if} \quad \operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

for μ-almost all $x \in X$.

Proof. Note that

$$\operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

if and only if

$$\left\langle \psi(x)\xi, \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\eta \right\rangle \text{ for all } \xi, \eta \in \mathcal{H}$$

if and only if

$$\left\langle \xi, \psi(x)^{*}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\eta \right\rangle \text{ for all } \xi, \eta \in \mathcal{H}$$

if and only if

$$\psi(x)^{*}\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

as required. □

Corollary 3.2.4. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a quantum random variable, then*

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0 \quad \textit{if and only if} \quad \mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

for all μ-almost all $x \in X$.

Proof. Since $\psi(x) \in \mathcal{B}(\mathcal{H})_+$ for all $x \in X$, it follows that $\psi(x) = \psi(x)^*$ implying that

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \psi^* \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.$$

The result is then immediate from the previous theorem. □

We are now able to prove that if the range of a quantum random variable is μ-almost everywhere orthogonal to the range of the principal Radon-Nikodým derivative then $\mathbb{E}_\nu [\psi] = 0$.

Theorem 3.2.5. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν-integrable quantum random variable with*

$$\mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

for μ-almost all $x \in X$, then

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = 0$$

for μ-almost all $x \in X$.

Proof. Notice that if

$$\operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

then by Lemma 3.2.2,

$$\operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right)$$

and by Theorem 3.2.3

$$\psi^* \frac{\mathrm{d}\nu}{\mathrm{d}\mu}^{1/2} = 0$$

so

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}^{1/2} \psi = 0.$$

Then, multiplying on the left by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}^{1/2}$$

gives

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

as desired. □

With positive quantum random variables, one is able to provide a more complete classification as the following theorems illustrate.

Theorem 3.2.6. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable, then $\mathbb{E}_\nu[\psi] = 0$ if and only if*

$$\psi(x)^{1/2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = 0$$

for μ-almost all $x \in X$.

Proof. Suppose that $\mathbb{E}_\nu[\psi] = 0$. Then

$$\int_X \operatorname{Tr}\left(\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right)\mathrm{d}\mu = \operatorname{Tr}(\rho\mathbb{E}_\nu[\psi]) = 0 \tag{3.2}$$

for every $\rho \in S(\mathcal{H})$. Since $\psi(x) \in \mathcal{B}(\mathcal{H})_+$, it follows that

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} \geq 0$$

for every $x \in X$, which combined with equation 3.2, implies that

$$\operatorname{Tr}\left(\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right) = 0 \tag{3.3}$$

for every $x \in X$ and $\rho \in S(\mathcal{H})$. However,

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = z^*z$$

where $z = z(x)$ given by

$$z(x) = \psi(x)^{1/2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}.$$

Now suppose that $\rho = I/d$, then Equation (3.3) implies that $\operatorname{Tr}(z^*z) = 0$, from which it follows that $z = 0$. That is

$$\psi(x)^{1/2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = 0$$

for every $x \in X$. Then

$$\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} = 0$$

for every $x \in X$ and $\rho \in S(\mathcal{H})$. Therefore,

$$\operatorname{Tr}(\rho\mathbb{E}_\nu[\psi]) = \int_X \operatorname{Tr}\left(\rho\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\psi(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right)\mathrm{d}\mu = 0$$

for every $\rho \in S(\mathcal{H})$ so that $\mathbb{E}_\nu[\psi] = 0$. $\square$

Corollary 3.2.7. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable, then*

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

if and only if

$$\psi(x)^{1/2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} = 0$$

for μ-almost all $x \in X$.

Proof. First suppose that

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0.$$

Then by Proposition 3.2.1 it follows that $\mathbb{E}_\nu [\psi] = 0$ and then by Theorem 3.2.6,

$$\psi(x)^{1/2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} = 0.$$

Now suppose that

$$\psi(x)^{1/2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} = 0$$

for μ-almost all $x \in X$. Then multiplying on the left by

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} \psi(x)^{1/2} = 0$$

gives

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} \psi(x)^{1/2} \psi(x)^{1/2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \right)^{1/2} = \left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right)(x) = 0$$

for μ-almost all $x \in X$ so that

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

as desired. □

Theorem 3.2.8. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable, then*

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0$$

if and only if $\mathbb{E}_\nu [\psi] = 0$.

Proof. Suppose that

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0.$$

Then by Corollary 3.2.4, it follows that

$$\mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right).$$

Now Theorem 3.2.5 implies that $\mathbb{E}_\nu [\psi] = 0$.

Similarly, suppose that $\mathbb{E}_\nu [\psi] = 0$. Then by Theorem 3.2.6,

$$\psi(x)^{1/2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} = 0$$

for every $x \in X$. Hence multiplying by $\psi(x)^{1/2}$ on the left and $\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}$ on the right implies

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0.$$

□

A similar equivalence holds for

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu},$$

which follows the same reasoning.

Finally, for general random variables, one is able to partially classify when they have mean zero.

Theorem 3.2.9. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν-integrable quantum random variable with*

$$\psi^* \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

then $\mathbb{E}_\nu [\psi] = 0$.

Proof. If

$$\psi^* \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

then Theorem 3.2.3 implies

$$\mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right).$$

Now Theorem 3.2.5 implies that $\mathbb{E}_\nu [\psi] = 0$. □

Corollary 3.2.10. *If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν-integrable quantum random variable with*

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0,$$

then $\mathbb{E}_\nu [\psi] = 0$.

Proof. From Theorem 3.2.9, if

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

then $\mathbb{E}_\nu [\psi^*] = 0$ so that

$$\mathbb{E}_\nu [\psi] = \mathbb{E}_\nu [\psi^{**}] = \mathbb{E}_\nu [\psi^*]^* = 0^* = 0,$$

which follows from Equation (2.8). □

Similarly

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

implies

$$0 = \psi^* \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} \psi^* \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}$$

and so

$$0 = 0^* = \left[\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2} \psi^* \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}\right]^* = \left[\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}\right]^* \psi^{**} \left[\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}\right]^* = \psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.$$

It is important to note that for a given quantum random variable ψ, $\mathbb{E}_\nu[\psi] = 0$ does not necessarily imply that

$$\psi \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0.$$

This notion is verified in the following example.

Example 3.2.11. Suppose $X = \{x_1, x_2\}$. Then it is sufficient to describe the action of ν and ψ on x_1 and x_2. Define

$$\nu(x_1) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad \nu(x_2) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

and

$$\psi(x_1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \psi(x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then since X is finite, the principal Radon-Nikodým Derivative is

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_j) = 2\frac{\nu(\{x_j\})}{\mathrm{Tr}(\nu(\{x_j\}))}$$

so that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1) = 2\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2) = 2\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1)\right)^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2)\right)^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \mathbb{E}_\nu\left[\psi\right] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 0. \end{aligned}$$

But,

$$\psi(x_1)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\psi(x_2)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In summary, we will collect our results for both positive valued quantum random variables and general quantum random variables in the following theorems.

Theorem 3.2.12. *Let $\psi : X \to \mathcal{B}(\mathcal{H})_+$ be a positive ν-integrable quantum random variable then the following statements are equivalent*

1. $\psi^*(x)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0$ *for μ-almost all $x \in X$*
2. $\mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$ *for μ-almost all $x \in X$*
3. $\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = 0$ *for μ-almost all $x \in X$*
4. $\psi^{1/2}(x)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{1/2}(x) = 0$ *for μ-almost all $x \in X$*
5. $\mathbb{E}_\nu[\psi] = 0$
6. $\psi(x)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0$ *for μ-almost all $x \in X$*

Proof. Let us first consider the following diagram that describes the relationships between the statements as described by earlier results.

$$\begin{array}{ccccccccc} 1 & \Longleftrightarrow & 2 & \Longrightarrow & 3 & \Longrightarrow & 5 & & \\ & & & & \Updownarrow & & \Updownarrow & & \\ & & & & 4 & & 6 & \Longleftrightarrow & 2 \end{array}$$

The related theorems for the above implications are collected in the following table.

Implication	$1 \iff 2$	$2 \implies 3$	$3 \iff 4$	$3 \implies 5$
Theorem	Theorem 3.2.3	Theorem 3.2.5	Corollary 3.2.7	Proposition 3.2.1

Implication	$5 \iff 6$	$2 \iff 6$
Theorem	Theorem 3.2.8	Corollary 3.2.4

$\square$

Theorem 3.2.13. *Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a ν-integrable quantum random variable then consider the following statements.*

1. $\psi^*(x)\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0$ *for μ-almost all $x \in X$*
2. $\mathrm{Ran}(\psi(x)) \perp \mathrm{Ran}\left(\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$ *for μ-almost all $x \in X$*
3. $\left(\psi \boxtimes \dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) = 0$ *for μ-almost all $x \in X$*
4. $\mathbb{E}_\nu[\psi] = 0$
5. $\psi(x)\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 0$ *for μ-almost all $x \in X$*

Then the following diagram describes the relationships between the statements as described in earlier results.

$$1 \iff 2 \implies 3 \implies 4 \impliedby 5$$

Moreover, the converses of 2 *implies* 3, 3 *implies* 4, *and* 5 *implies* 4 *do not hold in general.*

Proof. Let us first consider the following diagram that describes the relationships between the statements as described by earlier results.

$$1 \iff 2 \implies 3 \implies 4 \impliedby 5$$

The related theorems for the above implications are collected in the following table.

Claim	$1 \iff 2$	$2 \implies 3$	$3 \implies 4$	$4 \impliedby 5$
Theorem	Theorem 3.2.3	Theorem 3.2.5	Proposition 3.2.1	Corollary 3.2.10

Towards showing that 3 does not imply 2, suppose ψ is a quantum random variable and notice that if

$$\operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)$$

then by Lemma 3.2.2,

$$\operatorname{Ran}(\psi(x)) \perp \operatorname{Ran}\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right)$$

and by Theorem 3.2.3

$$\psi^* \frac{\mathrm{d}\nu}{\mathrm{d}\mu}^{1/2} = 0,$$

which holds if and only if

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}^{1/2} \psi = 0.$$

Towards this end, suppose $X = \{x_1, x_2\}$ and define

$$\nu(\{x_1\}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \nu(\{x_2\}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

as well as

$$\psi(x_1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \psi(x_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then notice that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and so

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1)\right)^{1/2} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2)\right)^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

Then it follows that

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1)\right)^{1/2} \psi(x_1) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2)\right)^{1/2} \psi(x_2) = 0$$

and

$$\psi(x_1)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1)\right)^{1/2} = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} \quad \text{and} \quad \psi(x_2)\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2)\right)^{1/2} = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

as desired.

To show that 4 does not imply 3, suppose that $X = \{x_1, x_2\}$ and $\nu(\{x_1\}) = \nu(\{x_2\}) = 1/2I_2$, where I_2 is the 2×2 identity matrix. Then suppose that $\psi(x_1) = I_2$ and $\psi(x_2) = -I_2$. Hence

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2) = I_2$$

so that

$$\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_1)\right)^{1/2} = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x_2)\right)^{1/2} = I_2.$$

Then

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x_1) = I_2 \quad \text{and} \quad \left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x_2) = -I_2$$

so that

$$\left(\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)(x) \neq 0$$

for $x \in X$. On the other hand,

$$\mathbb{E}_\nu\left[\psi\right] = I_2 - I_2 = 0$$

but

$$\psi \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \neq 0.$$

The fact that 4 does not imply 5 is Example 3.2.11 $\square$

Chapter 4

A Quantum Martingale Convergence Theorem

Our goal in this chapter is to prove the main result of this thesis, namely an operator valued version of the martingale convergence theorem for a particular class of operator valued martingales that we will define shortly. First, we will recall some of the well known classical results and discuss an important example. Second, we will then discuss some important features of quantum conditional expectation that will allow us to define a quantum martingale. Finally, we will prove that an analogue of the martingale convergence theorem holds for a specific quantum martingale. This martingale is of particular interest since it exhibits non-classical behaviour.

4.1 Classical Results

In this section, we will recall the classical definition of a martingale and we will state the martingale convergence theorem. We will also state an example, which will be investigated further in the section on quantum martingales. For further details, see [19].

First, let us recall the classical definition of a discrete time martingale.

Definition 4.1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process $\{M_j\}_{j\in\mathbb{N}}$ is called a martingale with respect to the filtration $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ if

1. M_j is $\mathcal{F}_j$-measurable for all $j \in \mathbb{N}$,
2. $\mathbb{E}|M_j| < \infty$ for all $j \in \mathbb{N}$, i.e. $M_j \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, and
3. $M_j = \mathbb{E}[M_{j+1}|\mathcal{F}_j]$.

We also know that under certain hypotheses a martingale $\{M_j\}_{j\in\mathbb{N}}$ converges to a random variable M_∞. The following convergence theorem is not the most general classical result, but it is sufficient for our purposes.

Theorem 4.1.2 (Classical Martingale Convergence Theorem)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose the stochastic process $\{M_j\}_{j\in\mathbb{N}}$ is a martingale with respect to the filtration $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$. Suppose further that $\mathbb{E}|M_j| < R$ for all $n \in \mathbb{N}$ with $R \in \mathbb{R}^+$. Then there exists a random variable M_∞ such that $\mathbb{E}|M_\infty| < \infty$ and $\{M_j\}_{j\in\mathbb{N}}$ converges to M_∞ almost surely.*

The following special case of the martingale convergence theorem is of particular interest since the quantum analogues will be investigated.

Example 4.1.3. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space with filtration $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$. Suppose further that $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$. Then the sequence $\{M_j\}_{j\in\mathbb{N}}$ defined by $M_j = \mathbb{E}[X|\mathcal{F}_j]$ is a martingale. The first condition for a martingale is satisfied trivially since M_j is, by definition of conditional expectation, $\mathcal{F}_j$-measureable.

In order to verify the second condition, notice that conditional expectation is defined so that

$$\int_E \mathbb{E}[X|\mathcal{F}_j]\,\mathrm{d}P = \int_E X\,\mathrm{d}P$$

for all $E \in \mathcal{F}_j$. In particular, since $\Omega \in \mathcal{F}_j$ and $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, we have

$$\int_\Omega M_j\,\mathrm{d}P = \int_\Omega \mathbb{E}[X|\mathcal{F}_j]\,\mathrm{d}P = \int_\Omega X\,\mathrm{d}P < \infty.$$

The third condition follows from the tower property, which is a well known result in classical probability. More precisely,

$$\mathbb{E}[M_{j+1}|\mathcal{F}_j] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_{j+1}]|\mathcal{F}_j] = \mathbb{E}[X|\mathcal{F}_j] = M_j.$$

In Theorem 4.2.2 we will establish a quantum version of the tower property.

Corollary 4.1.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ is a filtration and X is a random variable such that $\mathbb{E}|X| < \infty$. Then $M_j = \mathbb{E}[X|\mathcal{F}_j]$ converges to $M_\infty = \mathbb{E}[X|\mathcal{F}_\infty]$ where $\mathcal{F}_\infty = \sigma\left(\bigcup_{j=1}^{\infty} \mathcal{F}_j\right)$. If either*

1. *X is $\mathcal{F}_\infty$-measurable, or*

2. *$\mathcal{F}_\infty = \mathcal{F}$*

then $M_\infty = X$.

4.2 Quantum Conditional Expectation

This section collects new results that ensure the sensible definition of a quantum martingale is well defined.

The following proposition ensures that one is able to repeatedly apply quantum conditional expectation to a fixed quantum random variable. The subsequent proposition follows quickly from the results of [2].

Proposition 4.2.1. *Suppose that $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable and let $\mathcal{F}(X)$ be a sub σ-algebra of $\mathcal{O}(X)$. Define $\varphi = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$. Then φ is a ν-integrable quantum random variable, $\mathbb{E}_\nu\left[\varphi\right] \neq 0$, and $\varphi : X \to \mathcal{B}(\mathcal{H})_+$ for ν'-almost all $x \in X$ where $\nu' = \nu|_{\mathcal{F}(X)}$ is the restriction of ν to $\mathcal{F}(X)$.*

Proof. The fact that φ is a quantum random variable follows immediately from the definition of quantum conditional expectation. In order to see that $\mathbb{E}_\nu\left[\varphi\right] \neq 0$, notice

$$\mathbb{E}_\nu\left[\varphi\right] = \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]\right] = \mathbb{E}_\nu\left[\psi\right] \neq 0$$

In order to see that $\varphi : X \to \mathcal{B}(\mathcal{H})_+$ for ν'-almost all $x \in X$, notice that φ is a non-principal Radon-Nikodým derivative in that

$$\varphi = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu'}$$

where $\nu' = \nu|_{\mathcal{F}(X)}$, and

$$\tilde{\nu}(E) = \int_E \psi \,\mathrm{d}\nu' \tag{4.1}$$

for all $E \in \mathcal{F}(X)$. The result is realized since the non-principal Radon-Nikodým derivative is ν'-almost everywhere positive as seen in Theorem 2.6.1. $\square$

Since we wish to take the quantum conditional expectation of a random variable that is itself a quantum conditional expectation, say $\varphi = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$, we require that $\varphi : X \to \mathcal{B}(\mathcal{H})_+$. The previous proposition only ensures that $\varphi : X \to \mathcal{B}(\mathcal{H})_+$ for ν'-almost all $x \in X$. Since $\nu' = \nu|_{\mathcal{F}(X)}$ is the restriction of ν onto $\mathcal{F}(X)$, it follows that $\varphi : X \to \mathcal{B}(\mathcal{H})_+$ for ν-almost all $x \in X$ since all ν'-measure zero sets have ν-measure zero. Then it is sensible to make the association

$$\varphi(x) = \begin{cases} \dfrac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu'}(x), & \dfrac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu'}(x) \in \mathcal{B}(\mathcal{H})_+ \\ 0, & \text{otherwise} \end{cases}$$

so that $\varphi : X \to \mathcal{B}(\mathcal{H})_+$ for all $x \in X$ where $\tilde{\nu}$ is as in Equation (4.1). We now prove the tower property for quantum conditional expectation. Note that this was not considered in [2].

Theorem 4.2.2. *Suppose that $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$ and $\mathcal{G}(X)$ and $\mathcal{F}(X)$ are sub σ-algebras of $\mathcal{O}(X)$ such that $\mathcal{F}(X) \subset \mathcal{G}(X)$. Then*

$$\mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]|\mathcal{G}(X)\right] = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right] = \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi|\mathcal{G}(X)\right]|\mathcal{F}(X)\right].$$

Proof. Define $\varphi_f = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$ and $\varphi_g = \mathbb{E}_\nu\left[\psi|\mathcal{G}(X)\right]$. It will be shown that

$$\mathbb{E}_\nu\left[\varphi_f|\mathcal{G}(X)\right] = \varphi_f = \mathbb{E}_\nu\left[\varphi_g|\mathcal{F}(X)\right].$$

The first equality follows since $\mathcal{F}(X) \subset \mathcal{G}(X)$ and that φ_f is a $\mathcal{G}(X)$ measurable quantum random variable. Hence

$$\mathbb{E}_\nu\left[\varphi_f|\mathcal{G}(X)\right] = \varphi_f.$$

Now notice that for any $F \in \mathcal{F}(X)$ and for any $G \in \mathcal{G}$ that

$$\mathbb{E}_\nu\left[\varphi_f \chi_F\right] = \mathbb{E}_\nu\left[\psi \chi_F\right] \quad \text{and} \quad \mathbb{E}_\nu\left[\varphi_g \chi_G\right] = \mathbb{E}_\nu\left[\psi \chi_G\right].$$

But $F \in \mathcal{G}(X)$ since $\mathcal{F}(X) \subset \mathcal{G}(X)$. Therefore it follows that

$$\mathbb{E}_\nu \left[\varphi_g \chi_F\right] = \mathbb{E}_\nu \left[\psi \chi_F\right],$$

which implies that

$$\mathbb{E}_\nu \left[\varphi_f \chi_F\right] = \mathbb{E}_\nu \left[\varphi_g \chi_F\right]$$

for any $F \in \mathcal{F}(X)$ and therefore $\varphi_g = \varphi_f$. $\square$

4.3 A Quantum Martingale Convergence Theorem

In this section, we investigate the main results of this thesis. We give a definition of a quantum martingale and state an important example. We then prove an analogue of the classical Martingale Convergence Theorem holds for this specific martingale. Though the result resembles the classical theorem it is decidedly quantum, which provides an important relationship that is discussed in Section 3.2. A quantum version of the conditional dominated convergence theorem is also proved.

Definition 4.3.1. Let $(X, \mathcal{O}(X), \nu)$ be a quantum probability space. A sequence of quantum random variables $\{\varphi_n\}_{n\in\mathbb{N}}$ is called a quantum martingale with respect to the filtration $\{\mathcal{F}_n(X)\}_{n\in\mathbb{N}}$ if

1. φ_n is $\mathcal{F}_n(X)$-measurable for all $n \in \mathbb{N}$,
2. φ_n is ν-integrable for all $n \in \mathbb{N}$, and
3. $\mathbb{E}_\nu\left[\varphi_{n+1} | \mathcal{F}_n(X)\right] = \varphi_n$ for all $n \in \mathbb{N}$.

Theorem 4.3.2. *If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable and $\mathbb{E}_\nu\left[\psi\right] \neq 0$, then the sequence of $\mathcal{F}_n(X)$ measurable ν-integrable quantum random variables $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\varphi_n = \mathbb{E}_\nu\left[\psi | \mathcal{F}_n(X)\right]$ is a quantum martingale.*

Proof. The fact that φ_n is $\mathcal{F}_n(X)$ measurable follows immediately from the definition of conditional expectation. Similarly, the fact that φ_n is ν-integrable follows since

$$\mathbb{E}_\nu\left[\varphi_n\right] = \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi | \mathcal{F}_n(X)\right]\right] = \mathbb{E}_\nu\left[\psi\right]$$

and ψ is ν-integrable. Now notice that

$$\mathbb{E}_\nu\left[\varphi_{n+1} | \mathcal{F}_n(X)\right] = \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi | \mathcal{F}_{n+1}(X)\right] | \mathcal{F}_n(X)\right]$$

Then by the tower property, Theorem 4.2.2, it follows that

$$\mathbb{E}_\nu\left[\mathbb{E}_\nu\left[\psi | \mathcal{F}_{n+1}(X)\right] | \mathcal{F}_n(X)\right] = \mathbb{E}_\nu\left[\psi | \mathcal{F}_n(X)\right] = \varphi_n$$

which proves that φ_n is a martingale. □

Theorem 4.3.3 (Continuity of Quantum Conditional Expectation)**.** *Fix a sub σ-algebra $\mathcal{F}(X) \subset \mathcal{O}(X)$. Assume that the sequence of ν-integrable quantum random variables $\{\psi_n\}_{n\in\mathbb{N}}$ converges ultraweakly μ-almost surely to ψ. Suppose $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and assume that $\psi_n : X \to \mathcal{B}(\mathcal{H})_+$ and $\mathbb{E}_\nu\left[\psi_n\right] \neq 0$ for all $n \in \mathbb{N}$. Then $\mathbb{E}_\nu\left[\psi_n|\mathcal{F}(X)\right]$ converges to $\mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$ ultraweakly μ-almost surely.*

Proof. Notice that since ψ_n converges to ψ ultraweakly μ almost surely, it follows from Theorem 3.1.4 that $\mathbb{E}_\nu\left[\psi_n\right]$ converges to $\mathbb{E}_\nu\left[\psi\right]$ ultraweakly. Now suppose that $F \in \mathcal{F}$ and notice that since $\psi_n \to \psi$, it follows that $\psi_n\chi_F \to \psi\chi_F$, and so Theorem 3.1.4 says that $\mathbb{E}_\nu\left[\psi_n\chi_F\right] \to \mathbb{E}_\nu\left[\psi\chi_F\right]$. Therefore $\mathbb{E}_\nu\left[\psi_n|\mathcal{F}(X)\right] \to \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$ as required. □

There is an important relationship between $\varphi = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right]$ and the collection of complex random variables φ_ρ with $\rho \in S(\mathcal{H})$. This relationship will allow the use of the classical Martingale Convergence Theorem in the proof of the quantum version.

Proposition 4.3.4. *Suppose $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable with $\mathbb{E}_\nu\left[\psi\right] \neq 0$. Then*

$$\nu(\{x|\varphi(x) = \mathbb{E}_\nu\left[\psi|\mathcal{F}(X)\right](x)\}) = 1$$

if and only if

$$\mu(\{x|\varphi_\rho(x) = \mathbb{E}_\mu\left[\psi_\rho|\mathcal{F}(X)\right](x) \quad \forall\rho \in S(\mathcal{H})\}) = 1.$$

Proof. Suppose that $\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)]$. Then φ is a $\mathcal{F}(X)$-measurable quantum random variable such that for any $E \in \mathcal{F}(X)$,

$$\mathbb{E}_\nu[\varphi\chi_E] = \mathbb{E}_\nu[\psi\chi_E]$$

if and only if

$$\mathrm{Tr}(\rho\mathbb{E}_\nu[\varphi\chi_E]) = \mathrm{Tr}(\rho\mathbb{E}_\nu[\psi\chi_E])$$

if and only if

$$\mathbb{E}_\mu[(\varphi\chi_E)_\rho] = \mathbb{E}_\mu[(\psi\chi_E)_\rho]$$

if and only if

$$\mathbb{E}_\mu[\varphi_\rho\chi_E] = \mathbb{E}_\mu[\psi_\rho\chi_E]$$

where $(\varphi\chi_E)_\rho = \varphi_\rho\chi_E$ since on E, both functions give φ_ρ and off E they are both zero. Therefore it follows that $\varphi_\rho = \mathbb{E}_\nu[\psi_\rho|\mathcal{F}(X)]$. $\square$

It is important to note that conditional expectation is almost everywhere unique. Since the previous proposition relates the quantum conditional expectation and classical conditional expectation, one needs to be explicit about which measure is used.

We are about to prove the quantum martingale convergence for the quantum martingale

$$\varphi_n = \mathbb{E}_\nu[\psi|\mathcal{F}_n(X)].$$

Although we will prove that the sequence $\{\varphi_n\}$ has a unique limit, the value of the limiting random variable φ_∞ is not easily determined. All we are able to say is that φ_∞ and $\mathbb{E}_\nu\left[\psi|\mathcal{F}_\infty(X)\right]$ differ by a quantum random variable Φ such that $\Phi_\rho = 0$ for all $\rho \in S(\mathcal{H})$.

Theorem 4.3.5. *Let $(X, \mathcal{O}(X), \nu)$ be a quantum probability space and let $\psi : X \to \mathcal{B}(\mathcal{H})_+$ be a ν-integrable quantum random variable. Consider the quantum martingale $\varphi_n = \mathbb{E}_\nu\left[\psi|\mathcal{F}_n(X)\right]$. Then there exists a ν-integrable quantum random variable φ_∞ such that*

1. $\lim\limits_{n\to\infty} \varphi_n = \varphi_\infty$ *ultraweakly μ-almost surely,*
2. φ_∞ *is* $\mathcal{F}_\infty(X) = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)\right)$*-measurable, and*
3. $\varphi_\infty \in \{\mathbb{E}_\nu\left[\psi|\mathcal{F}_\infty\right] + \Phi \mid \Phi_\rho = 0 \ \forall \rho \in S(\mathcal{H})\}$.

Proof. Consider the sequence of complex random variables $\{\varphi_{n_\rho}\}$ with $\rho \in S(\mathcal{H})$. Then since φ_n is ν-integrable it follows that φ_{n_ρ} is μ-integrable by definition. Moreover,

$$\mathbb{E}_\mu \left|\varphi_{n_\rho}\right| = \mathbb{E}_\mu \left|\mathbb{E}_\mu\left[\psi_\rho|\mathcal{F}_n(X)\right]\right| \leq \mathbb{E}_\mu \left|\psi_\rho\right|$$

for all $n \in \mathbb{N}$. Therefore by the classical martingale convergence theorem, for every $\rho \in \mathrm{S}(\mathcal{H})$ there exists a $\tilde{\varphi}_{\infty_\rho}$ such that

1. $\lim_{n\to\infty} \varphi_{n_\rho} = \tilde{\varphi}_{\infty_\rho}$

2. $\tilde{\varphi}_{\infty_\rho}$ is $\mathcal{F}_\infty(X) = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)\right)$-measurable, and

3. $\tilde{\varphi}_{\infty_\rho} = \mathbb{E}_\mu\left[\psi_\rho | \mathcal{F}_\infty(X)\right]$.

But this implies that the sequence of quantum random variables φ_n converges to some φ_∞ ultraweakly μ-almost surely with $\varphi_{\infty_\rho} = \tilde{\varphi}_{\infty_\rho}$ for all $\rho \in S(\mathcal{H})$. Then by the continuity of quantum expectation, it follows that φ_∞ is ν-integrable.

Let

$$\tilde{\varphi} = \mathbb{E}_\nu\left[\psi | \mathcal{F}_\infty(X)\right]$$

so that

$$\tilde{\varphi}_\rho = \mathbb{E}_\mu\left[\psi_\rho | \mathcal{F}_\infty(X)\right] = \tilde{\varphi}_{\infty_\rho} = \varphi_{\infty_\rho}.$$

However, if Φ is another ν-integrable quantum random variable with $\Phi_\rho = 0$ for all $\rho \in S(\mathcal{H})$, then

$$\left(\tilde{\varphi} + \Phi\right)_\rho = \tilde{\varphi}_\rho + \Phi_\rho = \tilde{\varphi}_\rho = \varphi_{\infty_\rho}.$$

This implies

$$\varphi_\infty \in \{\mathbb{E}_\nu\left[\psi | \mathcal{F}_\infty\right] + \Phi \mid \Phi_\rho = 0 \ \forall \rho \in S(\mathcal{H})\}$$

as claimed. $\square$

Corollary 4.3.6. *If either (i) $\mathcal{F}_\infty(X) = \mathcal{O}(X)$ or (ii) ψ is $\mathcal{F}_\infty(X)$-measurable, then $\mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)] = \psi$ so that $\varphi_{\infty_\rho} = \psi_\rho$ and $\varphi_\infty \in \{\psi + \Phi \mid \Phi_\rho = 0\ \forall \rho \in S(\mathcal{H})\}$.*

Proof. Assuming the other hypothesis, the fact that $\psi = \mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)]$ follows from the definition of conditional expectation. □

We will now study the set of possible limits from our quantum martingale convergence theorem.

Theorem 4.3.7. *Let $(X, \mathcal{O}(X), \nu)$ be a quantum probability space and let $\psi : X \to \mathcal{B}(\mathcal{H})_+$ be a ν-integrable quantum random variable. Define the set*

$$\Gamma_{\nu,\psi} = \{\Psi | \Psi = \mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)] + \Phi \textit{ with } \Phi_\rho = 0\ \forall \rho \in S(\mathcal{H})\}.$$

If $\Psi_1 \in \Gamma_{\nu,\psi}$ then $\Psi_2 \in \Gamma_{\nu,\psi}$ if and only if

$$(\Psi_2 - \Psi_1) \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0.$$

Proof. Suppose $\Psi_1, \Psi_2 \in \Gamma_{\nu,\psi}$. Then $\Psi_{1_\rho} = \Psi_{2_\rho}$ for all $\rho \in S(\mathcal{H})$. Therefore,

$$\mathrm{Tr}\left(\rho\left(\Psi_1 \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right) = \mathrm{Tr}\left(\rho\left(\Psi_2 \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right),$$

which implies that

$$0 = \mathrm{Tr}\left(\rho\left(\Psi_2 \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right) - \mathrm{Tr}\left(\rho\left(\Psi_1 \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right)$$

and so

$$0 = \mathrm{Tr}\left(\rho\left((\Psi_2 - \Psi_1) \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right).$$

Then since this equality holds for all $\rho \in S(\mathcal{H})$, it follows that

$$(\Psi_2 - \Psi_1) \boxtimes \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 0$$

as desired.

Conversely, following the same reasoning in reverse gives the theorem. $\square$

We can now use our results from Section 3.2 to study $\Gamma_{\nu,\psi}$. We know that if Φ is a quantum random variable then $\Phi_\rho = 0$ implies $\mathbb{E}_\nu[\Phi] = 0$ whereas the converse is not necessarily true.

Corollary 4.3.8. *If* $\Sigma_{\nu,\psi} = \{\Psi | \Psi = \mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)] + \Phi,\ \mathbb{E}_\nu[\Phi] = 0\}$, *then* $\Gamma_{\nu,\phi} \subseteq \Sigma_{\nu,\psi}$.

Proof. Suppose that $\Psi \in \Gamma_{\nu,\psi}$. Then $\Psi = \mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)] + \Phi$ where $\Phi_\rho = 0$ for all $\rho \in S(\mathcal{H})$. Then by the earlier remark, it follows that $\mathbb{E}_\nu[\Phi] = 0$, so that $\Psi = \mathbb{E}_\nu[\psi|\mathcal{F}_\infty(X)] + \Phi$ with $\mathbb{E}_\nu[\Phi] = 0$. Hence $\Psi \in \Sigma_{\nu,\psi}$ and $\Gamma_{\nu,\psi} \subseteq \Sigma_{\nu,\psi}$. $\square$

Chapter 5

Conclusion and Future Work

In this thesis, we have extended the study of integration of operator valued functions with respect to positive operator valued measures first introduced in [3] and further studied in [2], to sequences of quantum random variables. In particular, we have proved a continuity of quantum expectation result for positive operator valued probability measures in Theorem 3.1.4. We have also introduced the concept of a quantum martingale with respect to a positive operator valued probability measure and in the case of a particular quantum martingale, we have proved a quantum version of the classical martingale convergence theorem, which is Theorem 4.3.5. This result is of particular interest since in the quantum setting, the limit is decidedly non-classical.

In future work, we would like to consider a more general quantum version of the martingale convergence theorem. There are a number of technical difficulties that need to be overcome, however, in order to prove such a theorem. In addition to considering a more general martingale convergence theorem, we would like to classify the current quantum martingale convergence theorem limit more completely. We conjecture that "removing the mean zero part" of a random variable $\Psi \in \Gamma_{\nu,\psi}$ will result in the unique limit that is described in the classical theorem. In order to do so, we wish to construct an inner product on the space of all quantum random variables and consider them as a Hilbert space $\mathbb{H}$, so that

$$\mathbb{H} = \Gamma_\nu \oplus \Gamma_\nu^\perp$$

where

$$\Gamma_\nu = \{\Phi | \Phi_\rho = 0 \; \forall \rho \in S(\mathcal{H})\}.$$

In the case where the underlying sample space X is finite, initial calculations suggest that there is a suitable inner product on $\mathbb{H}$ such that the projection of $\Psi \in \Gamma_{\nu,\psi}$ onto $\Gamma_\nu^\perp$ is indeed the unique limit $\phi_\infty = \mathbb{E}_\nu \left[\psi | \mathcal{F}_\infty(X)\right]$. However, it is not at all clear that this can be done when X is not finite.

In addition to fully understanding the martingale convergence theorem in the quantum setting, we would like to investigate other limiting properties of quantum conditional expectation with a family of quantum random variables. In particular, suppose that $I = [0,1]$ with the Borel sets $\mathcal{O}(I)$ and Lebesgue measure and consider the collection of quantum random variables

$$\psi : (I \times X, \mathcal{O}(I) \otimes \mathcal{O}(X)) \to (\mathcal{B}(\mathcal{H}), \mathcal{O}(\mathcal{B}(\mathcal{H})))$$

where $\mathcal{O}(I) \otimes \mathcal{O}(X)$ is the product σ-algebra. Then say that ψ is measurable if for every state $\rho \in S(\mathcal{H})$, the complex valued function $f_\rho : (I, X) \to \mathbb{C}$ given by

$$(t, x) \mapsto \mathrm{Tr}(\rho\psi(t, x))$$

is measurable. That is, $f_\rho^{-1}(E) \in \mathcal{O}(I) \otimes \mathcal{O}(X)$ for every $E \in \mathcal{O}(\mathbb{C})$. Assuming that ψ is measurable and integrable, it is natural to ask when $\psi(t, \cdot)$ is a ν-integrable quantum random variable for fixed $t \in I$ and when $\psi(\cdot, x)$ is a Lebesgue integrable function for fixed $x \in X$. If μ is Lebesgue measure on I, then when can one interchange integration? Namely, in what circumstances are

$$\int_X \int_I \psi(t, x)\, \mathrm{d}\mu\, \mathrm{d}\nu$$

and

$$\int_I \int_X \psi(t, x)\, \mathrm{d}\nu\, \mathrm{d}\mu$$

equal?

In the classical setting, conditional expectation is also integral in defining a Markov process. Thus, a natural question to ask is whether or not one is able to define a quantum Markov process using the quantum conditional expectation defined in [2]. If such a definition is possible, it would be informative to study the properties of those processes as one would do in the classical setting.

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