THE SELF-AVOIDING WALK

A REPORT

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Abstract

This report reviews the main and most recent results about the scaling limit of self-avoiding walk (SAW) on the integer lattice in dimensions greater than or equal to two.

There had been many efforts to obtain rigorous results and answers to these questions about the self-avoiding path: How many possible paths are there for the SAW? What would be the distance on average from the origin to the endpoint of the walk? What is the scaling limit of the self-avoiding walk when the number of steps N tends to infinity? However, exact results have been proved only for dimensions $d \geq 5$, leaving the findings about the other dimensions as conjectures. The objective of this report is to state these results and conjectures about the self-avoiding walk in all dimensions in an organized manner.

The first chapter of this work provides some brief background and history about the self-avoiding walk, the main definitions about stochastic processes and convergence that will be needed to analyze the scaling limit of the SAW as well as the main critical exponents and the connective constant.

Chapter 2 presents the conjectured scaling limit for the self-avoiding walk in dimension $d = 4$ and the most important result and theorem obtained for dimensions $d \geq 5$. In chapter 3, the necessary background in complex analysis is presented (such material can be found in a variety of texts) as well as the Loewner equation, the Schramm-Loewner evolution (SLE) and the conjectured scaling limit of the SAW in $d = 2$.

The final chapter introduces the recent research by Tom Kennedy on the test of SLE predictions for the two dimensional self-avoiding walk through Monte Carlo simulations which supports the conjecture made for dimension two.

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Dedications

To my Family, close people and professors that have always been there to help and encourage me to keep going, and most important to my mother who made me the successful person that I am.

Contents

Chapter 1

Introduction

The different applications of the simple random walk and its derived models have been of fundamental use in probability theory as well as in chemistry, physics, modeling, and other mathematical areas. This is essentially due to the advantages involved in working with discrete processes, such as the less complicated and simpler simulations and an easier understanding of the model. One important model derived from a simple random walk (SRW) is the self-avoiding random walk (SAW) which has been studied for nearly half a century and was developed initially in physical chemistry with the intention of modeling polymer chains when placed in a good solvent. Polymers, of course, have the unique characteristic that each chain cannot cross itself at any point. Thus, the SAW is a reasonable model of polymer chains since the chain cannot visit any site more than once.

The answer to the problem presented by physicists and chemists is to try to find a simplified model that captures the essential properties of these polymers. Initially, the "best" mathematical model that could encode the polymer's unique properties was a random walk. The random walk approximation for polymers was proposed 60 years ago by a German chemist called Kuhn who presented a model for which the mean squared end to end distance R^2 (which represents the polymer chain's length) grew as the squared root of the degree of polymerization N (i.e., $R \sim N^{1/2}$).

Years after this assumption was made it was proven incorrect by arguing that R (for polymers) grows faster than $N^{1/2}$. A new answer to the questions about this model was discovered by the Nobel laureate Flory[1] who suggested that while the random walk tends to trap itself, the monomers try to bounce away from each other. (This is the so-called excluded volume constraint.) Thus he derived that at equilibrium $R \sim N^{3/(2+d)}$ where $d = 1, 2, 3$ is the dimension in which the polymer "lives". Ever since Flory presented his solution in terms of the self-avoiding walk (SAW), physicists have been trying to verify his predictions, and mathematicians have been trying make his arguments rigorous. Significant non-rigorous progress was made by Edwards and de Gennes[2] but there are still dimensions d for which no exact solution has yet to be found.

Before computers were invented, the Japanese physicist Teramoto tried to make calculations by hand for $N \leq 9$. With the invention of computers, new progress was achieved by modeling random walk paths without self-intersections which was done

[1]Nobel prize in Chemistry in 1974 for his fundamental achievements, both theoretical and experimental, in the physical chemistry of macromolecules.

[2]Nobel prize in Physics in 1991 for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers.

on square and cubic lattices. In 1954 Wall, Hiller, Wheeler and in 1955 Rosenbluth tried to write programs to run simulations of self-avoiding walks but the probability of reaching length N before self-intersecting was minimal. In 1982, the physicist Nienhuis found an exact solution for a two dimension model similar to the Ising model, and assuming that this model and SAW are equivalent then the result presented by Flory is correct, but this assumption was never proved in a rigorous mathematical sense. Finally in the 1980's, Hara, Slade, Lawler, Schramm, Werner and other mathematicians in an attempt to establish rigorous results made significant progress which will be addressed in this report.

It is also important to mention that sometimes a continuous process can be introduced which is both easier to understand than the discrete process and shares the same critical exponents. For example, the discrete process simple random walk converges to Brownian motion (BM) in every dimension. In other words, the continuous process Brownian motion is the scaling limit of simple random walk.

1.1 Stochastic processes

Before more detail will be offered about the self-avoiding random walk (SAW), its scaling limits and behavior in all different dimensions, there are a number of necessary concepts, notions and fundamental ideas regarding stochastic processes that should be introduced.

Let the probability space be $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that Ω is a space, $\mathcal{F}$ is a σ-algebra,

$\mathbb{P}$ is the probability measure on $(\Omega, \mathcal{F})$, and $X : \Omega \to \mathbb{R}^d$ is a measurable function or random variable. Suppose that $I \subset \mathbb{R}^d$ is any indexing set of infinite cardinality and for each $\alpha \in I$ there is a random variable $X_\alpha : \Omega \to \mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

A stochastic processes $X = \{X_\alpha, \alpha \in I\}$ is a collection of random variables indexed by I. It is convenient to view I as time; two cases of these processes are considered, the first one is if $I = \mathbb{Z}^+$ in which case we consider the discrete time stochastic process $S = \{S_n,\, n = 0, 1, 2, 3, \ldots\}$ which is a countable collection of random variables indexed by the non-negative integers, and the second is if $I = [0, \infty)$ in which case we consider the continuous time stochastic process $B = \{B_t,\, 0 \leq t < \infty\}$ which is an uncountable collection of random variables indexed by the non-negative real numbers.

Now let $\mathbb{Z}^d = \{(z_1,\, z_2,\, \ldots,\, z_d) \,:\, z_i \in \mathbb{Z}\}$ be the d-dimensional integer lattice, let $e_1,\, e_2,\, \ldots,\, e_{2d}$ represent the $2d$ unit vectors in $\mathbb{Z}^d$ and let $X_1,\, X_2,\, \ldots$ be independent identically distributed (*i.i.d.*) random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $X_i : \Omega \to \mathbb{Z}^d$, $i = 1,\, 2,\, \ldots$ with probability

$$P\{X_i = e\} = \frac{1}{2d}, \;\; |e| = 1. \tag{1.1}$$

Definition 1.1. A discrete time stochastic process $S = \{S_n,\, n = 0, 1, 2, 3, \ldots\} = \{S_n,\, n \geq 0\}$ with $S_0 = x$ and

$$S_n = \sum_{i=1}^{n} X_i, \;\; n \geq 1,$$

is called a simple random walk starting at $x \in \mathbb{Z}^d$. Let its probability distribution be given by $P_n(x,\, y) = P^x\{S_n = y\}$. Note that S has stationary and independent increments.

Definition 1.2. A continuous time stochastic process $B = \{B_t,\, 0 \leq t < \infty\}$ is called a (one-dimensional, standard) Brownian motion on $\mathbb{R}$ if:

1. $B_0 = 0$,
2. $B_t - B_s$ is independent of B_s for $0 \leq s \leq t < \infty$,
3. $B_t - B_s \sim N(0,\, t - s)$ for $0 \leq s \leq t < \infty$,
4. the trajectory $t \mapsto B_t$ is continuous almost surely.

We say that B is a d-dimensional Brownian motion if $B = (B^1, \ldots, B^d)$ where the components B^j are independent, one-dimensional Brownian motions.

Brownian motion is a mathematical model which is fundamental in theory and applications of probability and originated by the several attempts to describe the movement of very small particles suspended in a fluid. This model has been used to analyze many other phenomena; the observed fluctuations of the stock market would be one example. Moreover, recalling what was mentioned before, Brownian motion is known to be the scaling limit of simple random walk (as well as several other stochastic processes), a fact that is closely related to the universality of the normal distribution. Actually, being the scaling limit of simple random walk means that as the size of the increments in a random walk tend to zero and the number of steps increases, it converges to a Brownian motion in the distributional sense. (We will discuss this in greater detail in the next section.) To be more precise, if the random walk step size is ε, one needs to take a walk of diameter L/ε^2 to approximate a Brownian motion of diameter L; similarly, this notion holds for all dimensions. Therefore, it can be shown that the convergence of simple random walk to Brownian motion is controlled by the Central Limit Theorem (CLT). This theorem states that after a large number of independent steps in the random walk, the random walk is distributed approximately normally with variance $\sigma^2 = \frac{t}{\Delta t}\varepsilon^2$, where t is the time that passed since the simple random walk started, ε is the step size as mentioned previously, and Δt is the time elapsed between two successive steps.

1.2 Convergence in distribution

Although the idea of convergence in distribution has been already mentioned, we will now present the formal definitions according to Protter and Jacod [6].

Let P_n and P be some probability measures on $\mathbb{R}^d$ $(d \geq 1)$. The sequence P_n converges weakly to P if $\int f(x)P_n(dx)$ converges to $\int f(x)P(dx)$ for each function f which is continuous and bounded on $\mathbb{R}^d$. That is,

$$P_n \to P \text{ weakly iff } \lim_{n\to\infty} \int f(x)P_n(dx) = \int f(x)P(dx)\ \forall f \in C_b(\mathbb{R}^d). \tag{1.2}$$

Definition 1.3. Let $\{X_n,\, n \geq 1\}$ be $\mathbb{R}^d$-valued random variables. We say X_n converges in distribution to X if the distribution measure P^{X_n} converges weakly to P^X where $P^{X_n}(A) = P(X_n \in A)$ for A Borel and similarly for P^X. We write $X_n \xrightarrow{D} X$.

The next theorem is standard and gives one characterization of convergence in distribution.

Theorem 1.4. *Let $\{X_n,\, n \geq 1\}$ be $\mathbb{R}^d$-valued random variables. Then $X_n \xrightarrow{D} X$ if and only if*

$$\lim_{n\to\infty} E\{f(X_n)\} = E\{f(X)\} \tag{1.3}$$

for all continuous and bounded functions f on $\mathbb{R}^d$.

The following theorem characterizes weak convergence and can be considered as complementary to the previous one.

Theorem 1.5 (Portmanteau Theorem). *If P_n and P are probability measures on the space $(\mathbb{R}, \mathcal{B})$, then the following results are equivalent.*

1. *$P_n \to P$ weakly as $n \to \infty$.*
2. *$\int_{\mathbb{R}} f(x)P_n(dx) \to \int_{\mathbb{R}} f(x)P(dx)$ as $n \to \infty$ for all $f \in C_b$ (where C_b is the set of all bounded and continuous functions on $\mathbb{R}$).*
3. *$\limsup_{n\to\infty} P_n(F) \leq P(F)$ for all closed sets $F \subset \mathbb{R}$.*
4. *$\liminf_{n\to\infty} P_n(G) \geq P(G)$ for all open sets $G \subset \mathbb{R}$.*
5. *$\lim_{n\to\infty} P_n(B) = P(B)$ for all Borel sets B with $P(\partial B) = 0$ (where ∂B is the boundary of B).*

We include an example to show that the inequalities in the Portmanteau Theorem can be strict.

Example 1.6. Let $\Omega = [0,\, 1]$ be the space, let $\mathcal{F}$ be the Borel sets of $[0,1]$, let $x_n = 1 - \frac{1}{n}$, and let the probability measures P_n and P be defined as follows:

$$P_n(B) = \begin{cases} 1, & \text{if}\, x_n \in B, \\ \\ 0, & \text{if}\, x_n \notin B, \end{cases} \quad \text{and} \quad P(B) = \begin{cases} 1, & \text{if}\, 1 \in B, \\ \\ 0, & \text{if}\, 1 \notin B, \end{cases}$$

for $B \in \mathcal{F}$. We claim that $P_n \to P$ weakly.

Proof. Let $f \in C_b(\mathbb{R})$ be continuous and bounded. Then,

$$\int_{\mathbb{R}} f(x) P_n(dx) = f(x_n) = f(1 - 1/n) \quad \text{and} \quad \int_{\mathbb{R}} f(x) P(dx) = f(1).$$

Since the function f is continuous, and since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (1 - \frac{1}{n}) = 1,$$

we conclude that

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(1).$$

In other words, $P_n \to P$ weakly. □

We will now show by example that strict inequalities hold in parts 3 and 4 of the Portmanteau Theorem.

Let $F = \{1\}$ which is a closed set. Therefore, $P_n(F) = 0$ for all n since $x_n = 1 - 1/n \notin F$. On the other hand, $P(F) = 1$ since $1 \in F$ and so we conclude

$$\limsup_{n\to\infty} P_n(F) = 0 < P(F) = 1.$$

Let $G = (0,\, 1)$ which is an open set. Therefore, $P_n(G) = 1$ for all n since $x_n = 1 - 1/n \in (0,\, 1)$. On the other hand, $P(G) = 0$ since $1 \notin (0,\, 1)$ and so we conclude

$$\liminf_{n\to\infty} P_n(G) = 1 > P(G) = 0.$$

The notion of weak convergence presented in (1.2) and (1.3) is fundamental for the results presented in this report. Note that this really is a weak form of convergence since what matters is that the probability distributions of the random variables are converging, and not the actual values of those random variables.

Following the definition and theorems of weak convergence and applying this concept to the relation between a simple random walk and a Brownian motion, one can state the following well-known theorem.

Theorem 1.7 (Donsker's Theorem). *If S is a simple random walk in $\mathbb{Z}^d$ and if*

$$X_n(t) = (d/n^{1/2})\{S([nt]) + (nt - [nt])(S([nt]) + 1) - S([nt]))\} \tag{1.4}$$

then $X_n \overset{D}{\to} X$ where X is a d-dimensional Brownian motion.

1.3 The self-avoiding walk (SAW)

Now that the basic definitions and notions about the stochastic processes have been provided, the main subject matter will be reexamined. The SAW is a model of profound significance in combinatorial probability theory, statistical physics and as mentioned previously in polymer chemistry; although it is a model of random walk paths it cannot be described in terms of transition probabilities. Consequently it is not a stochastic process[3] and as a result it is a model that is more difficult to analyze.

Definition 1.8. A self-avoiding walk (SAW) of length N in the d-dimensional lattice $\mathbb{Z}^d$ starting at x, is defined as a path $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ with $\omega_j \in \mathbb{Z}^d$, $\omega_0 = x$, $|\omega_j - \omega_{j-1}| = 1$, $j = 1, 2, \ldots, n$; and $\omega_i \neq \omega_j$ for $i \neq j$, $0 \leq i < j \leq n$. Let $|\omega| = N$ denote the length of ω. In other words, a SAW is a random walk path which does not visit the same site more than once.

When analyzing the self-avoiding walk, two important questions arise: How many possible paths are there for a self-avoiding walk? And, assuming each path is as likely as the other ones, what would be the distance on average from the origin to the point x? However, there is a third question that arises within the last two: What is the asymptotic behavior of the self-avoiding walk as N (steps in a self-avoiding walk) tends to infinity? One would think that the simplest way to answer the questions would be by using computer simulations, but several works done in this field have shown that due to the exponential growth of the number of paths as N increases, obtaining results for large N is almost impossible; thus, the exact counting of the possible paths as mentioned by Madras and Slade [20] has only been done for $N \leq 34$ in $d = 2$ and for $N \leq 21$ in $d = 3$.

The above questions must be asked in each dimension d in order to generalize the results. The easiest case and therefore the one with a trivial answer is that

[3] Although there are some who would classify SAW as a stochastic process, according to Slade [26, page 7], it is not.

for dimension $d = 1$. Indeed, a self-avoiding walk in one dimension has no other alternative but to move in the same initially chosen direction. Hence there only exists two paths for every value of N (recall that N is the number of steps in a self-avoiding walk), and therefore, the maximum distance for the origin is exactly N. In addition, it can be shown that higher dimensions ($d \geq 5$) provide a richer and more complex answer to the presented questions about SAW. Although it is also important to mention that the most interesting questions remain open for the low dimension ($d = 2, 3, 4$) cases.

As it will be explained throughout this report, the upper critical dimension for the self-avoiding walk above which all critical exponents are dimensional independent is $d = 4$ due to the fact that the random walk paths tend to intersect below four dimensions, and have an opposite behavior above it. Recall that as pointed out previously, there does not exist any rigorous proved results for the lower dimension cases ($d = 2, 3, 4$).

It seems clear that in high dimensions, the SAW should be closer to the simple random walk (we say that the mean field model for the self-avoiding walk is the SRW), provided that a simple random walk is less likely to intersect itself in higher dimensions (for $d > 4$). Hence, using rigorous mathematical analysis and in some of the cases high-precision computer simulations, the following has been concluded: for dimensions $d > 4$, the lace expansion has been used to prove the existence of answers and to solve the questions that were stated previously. Additionally, partial results for the case $d = 4$ have been obtained by applying logarithmic corrections. In contrast to dimensions four and above, the three dimensional case $d = 3$ remains mathematically unsolved. Finally, both rigorous and non-rigorous solutions for the two-dimensional case $d = 2$ (achieved and supported by numerical Monte Carlo simulations) have been found by associating it to the stochastic Loewner evolution (SLE).

In summary, assuming that the scaling limit is the law of the path $n^{-\nu}\omega$ when $n \to \infty$, where ω is the N-step self-avoiding walk, and supposing that the limit exists and is conformally invariant, it has been conjectured to be $\mathrm{SLE}_{8/3}$ for $d = 2$; it is not understood for $d = 3$; for the case of $d = 4$, using the logarithmic correction factor $[\log N]^{1/4}$, the scaling limit is believed to be Brownian motion; and finally, for dimensions $d \geq 5$ the lace expansion has been of fundamental use to demonstrate that the corresponding scaling limit is Brownian motion. It should be remarked that for the path $n^{-\nu}\omega$, the variable ν is one of the most important critical exponent.

In order to discuss the average distance measure from the origin to x after N-steps, we shall introduce a probability measure on the N-step SAW. This probability measure is the uniform measure which assigns equal probabilities to each N-step self-avoiding walk.

Definition 1.9. Let P_n denote the uniform probability measure on Γ_N, the set of all N-step self-avoiding walks starting at 0. That is, if $C_N = |\Gamma_N|$ is the cardinality of the set Γ_N, then the measure P_n is given by

$$P_n(\omega) = \frac{1}{C_N}, \ \ \forall\, \omega \in \Gamma_N. \tag{1.5}$$

1.3.1 The critical exponents ν and γ

It is believed that for the self-avoiding walk, there exists an exponential growth of C_N with some power law corrections, unlike the simple exponential growth of the SRW, where $(2d)^N$ is the exact number of N-step simple random walks. It is also believed that the mean-squared displacement, which gives the average distance (squared) from the origin x after N steps, will not always be linear in this number of steps, which is contrary to what is known for the simple random walk where the mean squared displacement is N. In fact, in d dimensions,

$$E|S_N|^2 = \frac{N}{d}.$$

According to Madras and Slade [20], the conjectured behavior of C_N and the mean-squared displacement $E^{P_n}[|\omega(N)|^2]$, are respectively:

$$C_N \sim A\mu^N N^{\gamma-1} \tag{1.6}$$

and

$$E^{P_n}[|\omega(N)|^2] \sim DN^{2\nu} \tag{1.7}$$

where $E^{P_n}[\cdot]$ is the expectation with respect to the uniform measure P_n, the values A, μ, D, γ and ν are all positive constants with dependence on the dimension that is being working in. Additionally, μ is referred to as the connective constant, and γ as well as ν are known as critical exponents (which are the exponents given by a power law equation).

Up until now, equations (1.6) and (1.7) have already been proved to hold for dimensions $d \geq 5$, but in dimension four, they must be modified by a logarithmic factor, and thus, the conjectured corrections to C_N and $E^{P_n}[|\omega(N)|^2]$ for dimension $d = 4$ given by Madras and Slade [20] are:

$$C_N \sim A\mu^N [\log N]^{1/4} \tag{1.8}$$

and

$$E^{P_n}[|\omega(N)|^2] \sim D[\log N]^{1/4}. \tag{1.9}$$

It is important to point out that for ordinary simple random walk, equations (1.6) and (1.7) hold with $\gamma = 1$ and $\nu = 1/2$, both for the nearest neighbor and more general walks.

The critical exponent that is related to the conjectured behavior of $E^{P_n}[|\omega(N)|^2]$ is called ν and its conjectured values depending on the dimension d are:

$$\nu = \begin{cases} 1, & d = 1, \\ 3/4, & d = 2, \\ .5888, & d = 3, \\ 1/2, & d = 4, \\ 1/2, & d \geq 5. \end{cases}$$

Furthermore, an earlier conjectured equation for the values of ν was made by the chemist Flory, who developed an effective but non-rigorous method for computing the exponent's values. The Flory exponents are given by $\nu_{Flory} = 3/(2+d)$ for $d \leq 4$ and $\nu_{Flory} = 1/2$ for $d > 4$. These equations agree with the conjectured values already presented for $d = 2$ and $d \geq 4$ (apart from the logarithmic correction needed for dimension $d = 4$), and comes very close for $d = 3$. However, the exact Flory value $\nu_{Flory} = 3/5$ in three dimensions has been proved non exact by numerical simulations.

It should be remarked that $\nu = 1$ in $d = 1$ is trivial. The result $\nu = 1/2$ as mentioned by Madras and Slade [20] has been proved in $d \geq 5$. This was to be expected since the self-avoiding walk moves away from the origin quicker than the simple random walk. However, that tendency should become less pronounced as the dimension increases. For this reason, it is expected that the critical exponent ν is a non increasing function of the dimension d (which is the most interesting characteristic of the self-avoiding model), but on the other hand it is independent of the type of allowed steps; this lack of dependence on the detailed definition of the model is known as universality.

The other important critical exponent is called γ, and besides characterizing the asymptotic behavior of C_N, it can be related to the probability that two N-step SAW starting at the same point x do not intersect each other. The conjectured values for γ, recalling that the only strict results confirming these values are for $d \geq 5$, and that the value for dimension $d = 4$ is obtained using a logarithmic correction, are the following:

$$\gamma = \begin{cases} 43/32, & d = 2, \\ 1.162, & d = 3, \\ 1, & d = 4, \\ 1, & d \geq 5. \end{cases}$$

In addition, using the estimate $d \leq \mu \leq 2d - 1$ we obtain $C_N \geq \mu^N$ for $N \geq 1$ that implies $\gamma \geq 1$ in all dimensions, although strictly speaking it is not known that γ exists, and there is no proof that γ is finite in $d = 2$, 3, or 4, where the best bounds are:

$$C_N \leq \begin{cases} \mu^N \exp[KN^{1/2}], & d = 2, \\ \mu^N \exp[KN^{2/(2+d)} \log N], & d = 3, 4. \end{cases} \tag{1.10}$$

It is important to mention that using an additional critical exponent called η, which gives the conjectured long distance behavior of the 2-point function $G_z(x, y) = \sum_{\omega : x \to y} z^{|\omega|}$ at the critical point $z_c = \frac{1}{\mu}$, we can obtain an equation that shows the relationship between the critical exponents ν, γ and η. This equation is called Fisher's scaling relation and is

$$\gamma = (2 - \eta)\nu. \tag{1.11}$$

So far we have introduced the critical exponents γ and ν, although we must remark that there are other critical exponents for which detailed definitions and notions will not be mentioned in this report. For further details see Madras and Slade [20].

1.3.2 The connective constant μ

It can be easily shown that $d^N \leq C_N \leq 2d(2d-1)^{N-1}$. Indeed the lower bound follows from only allowing steps in positive coordinate directions and the upper bound follows from disallowing immediate reversals. From this it follows that the first step in order to justify the equations (1.6), (1.8) and (1.10), if they correctly represent the behavior of C_N for large N, is to prove the existence of the connective constant μ, which is done by showing that the following limit exists:

$$\mu = \lim_{N\to\infty} (C_N)^{1/N}. \tag{1.12}$$

This limit does exist and it is bounded by $d \leq \mu \leq 2d-1$. The proof is easy and is provided by Madras and Slade [20]. Even though the exact value for the connective constant μ remains unknown for any dimension $d \geq 2$, using the lace expansion (this method will not be treated in this report) for high dimensions $d \geq 5$, Hara and Slade [4] showed that as $d \to \infty$,

$$\mu = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} + O(\frac{1}{(2d)^3}). \tag{1.13}$$

Recent studies completed by Slade [26] give an improved version of the value of the connective constant μ as follows:

$$\mu = 2d-1-\frac{1}{2d}-\frac{3}{(2d)^2}-\frac{16}{(2d)^3}-\frac{102}{(2d)^4}-\frac{729}{(2d)^5}-\frac{5533}{(2d)^6}-\frac{42229}{(2d)^7}-\frac{288761}{(2d)^8}+O(\frac{1}{(2d)^9}).$$

We say that the connective constant μ represents the effective coordination number of the considered lattice. Moreover, it depends on the number of steps N as well as on the dimension d that we are working in. Thus, μ is not universal.

Further information about the connective constant, critical exponents and specific facts and theorems mentioned in this section may be found in Madras and Slade [20].

Chapter 2

SAW in higher dimensions

2.1 SAW for $d = 4$

The dimension $d = 4$ is referred to as the upper critical dimension, above which normal or Gaussian behavior is observed. One argument use to predict $d = 4$ as the upper critical dimension is that Brownian motion paths or trajectories are two dimensional, and pairs of two-dimensional sets generically do not intersect above $4 = 2+2$ dimensions. This suggests that there is enough space to move in dimensions higher than 4 and that the self-avoiding restriction becomes non important, and thus, the self-avoiding walk should have the same Brownian scaling limit as a simple random walk.

As mentioned previously, dimension $d = 4$ can be seen as a separating line between the behavior of the self-avoiding walk for $d > 4$ ($\nu = 1/2$) and its behavior for $d < 4$ ($\nu > 1/2$). This result should and will be signaled by the appearance of a logarithmic correction in the assumptions for dimension four (for more details see [29]).

The detailed conjectured predictions for this logarithmic correction, which can be seen in equations (1.8) and (1.9) have been made by extending calculations through the non-rigorous renormalization group method which will not be further analyzed in this report.

Therefore, it is conjectured that for dimension $d = 4$, the scaling limit of self-avoiding walk (SAW) is believed to be Brownian motion. Recall that this limit, assuming it exist, is the distribution of the limit as $n \to \infty$ of the trajectory $[\log n]^{1/4} n^{-\nu}\omega$, where the logarithmic correction is made by using the following factor:

$$[\log n]^{1/4}. \tag{2.1}$$

2.2 SAW for $d \geq 5$

In this section, we state the principal theorem that provides the main result for the convergence of self-avoiding walks (SAW) in $d \geq 5$.

Recall that the details about self-avoiding walk (SAW) and convergence, and the bases for the understanding the next theorem were given in the first chapter.

To precisely state the main result provided by Madras and Slade [20], it first will be necessary to introduce this last notation.

Let $C_d[0, 1]$ denote the bounded continuous $\mathbb{R}^d$-valued functions on $[0, 1]$, equipped with the supremum norm. Given an N-step self-avoiding walk ω, let $X_n \in C_d[0, 1]$ by setting $X_n(k/n) = (Dn)^{1/2}\omega(k)$ for $k = 0, 1, 2, \ldots, n$, where D is the so-called diffusion constant, and taking $X_n(t)$ to be the linear interpolation of this. The Wiener measure on $C_d[0,\, 1]$ is denoted by dW, and $E^{P_n}[\cdot]_n$ denotes the expectation with respect to the uniform measure on the N-step self-avoiding walk (SAW). The main result is the following theorem presented by Madras and Slade [20].

Theorem 2.1. *For all $d \geq 5$, the scaled self-avoiding walk converges in distribution to a Brownian motion. In other words, for any bounded continuous function f on $C_d[0,\, 1]$,*

$$\lim_{n\to\infty} E^{P^{X_n}}[f(X_n)]_n = \int f dW. \tag{2.2}$$

In other words, this theorem states that if a self-avoiding walk does converge weakly, it must converge to a Brownian motion. Thus, the finite dimensional distributions of scaled SAW converge in distribution to those of Brownian motion, for a sufficiently large dimension ($d \geq 5$).

Using Theorem 1.4, we can restate this result as follows.

Theorem 2.2. *For all $d \geq 5$, $\nu = 1/2$ and $\gamma = 1$. Moreover, the distribution of $X_n(t,\, \omega) = n^{1/2}\omega([nt])$ under the uniform probability measure P_n, converges to a Gaussian or normal distribution as n approaches infinity.*

Although the rigorous proof of Theorem 2.1 [25] will not be included in this report, the main ideas involved in the proof concerning the lace expansion (introduced and derived by Brydges and Spencer involving an expansion and re-summation procedure) will be briefly introduced. For further details about the lace expansion we refer to Slade [26].

Let $Z_c = 1/\mu$ be the ratio of convergence of the power series defining the generating function of C_N for every $x \neq 0$, if $|Z| < Z_c$ and since $C_N(0, x) \leq C_N$ the 2-point function has ratio of convergence of at least Z_c then we will call Z_c the critical point. Furthermore, let

$$U_{st}(\omega) = \begin{cases} -1, & \omega(s) = \omega(t) \\ 0, & \omega(s) \neq \omega(t) \end{cases} \tag{2.3}$$

Define

$$K_\tau[a, b] = \prod_{(st)\in\Re_\tau[a, b]} (1 + U_{st}(\omega))$$

where $\Re_\tau[a, b] = \{(st) : 0 < t - s \leq \tau,\ s,\ t \in [a, b] \cap \mathbb{Z}\}$ is the set of all pair of integers (s, t) with $a \leq s < t \leq b$ for $\tau \geq 0$.

Now, the generating function $\chi(z)$ of C_N and the 2-point function $G_z(x, y)$ will be defined as follows:

$$\chi(z) = \sum_{N=0}^{\infty} C_N z^N \geq \sum_{N=0}^{\infty} (Mz)^N = \frac{1}{1 - \mu z} = \frac{1}{1 - z/z_c} \tag{2.4}$$

and

$$G_z(x, y) = \sum_{N=0}^{\infty} C_N(x, y) z^N \tag{2.5}$$

which are continuous at Z_c since $\chi(z) \to \infty$ as $Z \nearrow Z_c$.

We now introduce the variable Π_z which is considered the measure of the difference between the self-avoiding walk (SAW) and a simple random walk,

$$\Pi_z(0, v) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(0, v). \tag{2.6}$$

Hence, as a consequence of (2.6) we define the lace expansion as the expansion of $\Pi_z(0, v)$ which is the only method that has led to rigorous results about the scaling limit of self-avoiding walk (SAW) in higher dimensions ($d \geq 5$).

For further details, specific examples and demonstrations concerning the last results in dimensions greater than or equal to five we refer the reader to Hara and Slade [3], [4] and [5].

Chapter 3

SAW in $d = 2$

The SAW is one of the models used in statistical physics that is predicted to have a conformally invariant scaling limit in dimension $d = 2$, which would make it possible to obtain the values of the principal critical exponents ν and γ.

Oded Schramm in recent investigations presented a 2-dimensional conformally invariant random process that is called the stochastic Loewner evolution (now named the Schramm-Loewner evolution or SLE) [23]. This process depends on the parameter κ, and therefore is written SLE_κ. Moreover for different values of κ the process is related to the scaling limit of various models in dimension $d = 2$. In particular, Lawler, Schramm and Werner among others conjectured that $\mathrm{SLE}_{8/3}$ in a simple connected domain $D \subsetneqq \mathbb{C}$, provides the scaling limit for the self-avoiding walk in two dimensions (see [17] and [19] for more details).

3.1 Review of complex analysis

To completely understand the process and concepts involved in the Schramm-Loewner evolution and its application to the scaling limit of the SAW, it is fundamental to introduced a number of definitions and notions from complex analysis (for more details refer to [1] or [2]).

Recall that a complex number is defined as the linear combination of a real number component a and an imaginary component ib, where i is the imaginary unit with the characteristic that $i^2 = -1$, and b is a real number, so that $a+ib$ is a complex number.

We say $D \subset \mathbb{C}$ is a simply connected domain if D is an open, connected set larger than a single point in the complex plane $\mathbb{C}$ such that $D^c = \mathbb{C} \setminus D$ is connected

We define a complex valued function $f : D \to \mathbb{C}$ for a complex variable $z = a + ib$ to be differentiable at the point $z_0 \in D$ if the following derivative exists (see Duren [2]):

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{3.1}$$

Now, assuming $f : D \to \mathbb{C}$ to be a complex valued function:

1. If $f'(z_0)$ exists for every $z_0 \in D$, then f is *analytic* on D.

2. If $f(z_0) \neq f(z_1)$ for every z_0, $z_1 \in D$, then f is one to one on D, and is often called *univalent*.

Moreover, we call $f : D \to \mathbb{C}$ conformal if f is analytic and univalent on D. In particular, $f'(z_0) \neq 0 \, \forall \, z_0 \in D$.

Another important concept is the following normalization condition for the function $f : D \to \mathbb{C}$ known as the *hydrodynamic normalization*:

$$\lim_{z\to\infty,\, z\in D}(f(z) - z) = a, \tag{3.2}$$

where $a \geq 0$ is a non-negative real number.

3.1.1 Conformal mapping

Let D and D' be simply connected domains. Then, the function $f : D \to D'$ is called a *conformal transformation* if it is conformal on D and onto D', and consequently $f^{-1} : D' \to D$ is also a conformal transformation.

In 1851, Riemann enunciated one of the most important theorems in complex analysis: the fact that every simply connected domain other than the whole complex plane $\mathbb{C}$ itself can be mapped conformally onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The theorem as presented by Ahlfors [1] is as follows.

Theorem 3.1 (Riemann Mapping Theorem). *Given any simply connected domain D which is not the whole plane, and a point $z_0 \in D$, there exist a unique conformal transformation $f : D \to \mathbb{D}$, normalized by the two conditions $f(z_0) = 0$ and $f'(z_0) > 0$.*

Furthermore, there is an important fact derived from the Riemann mapping theorem, which is that the only conformal mappings $f : \mathbb{D} \to \mathbb{D}$ are the *Möbius transformations*:

$$f(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0 \tag{3.3}$$

where $a,\ b,\ c,\ d$ are complex constants.

If D is a Jordan domain (the interior of simple closed curve), Theorem 3.1 can be extended continuously to the boundary, so that the extended function will map the boundary curve in one to one fashion onto the unit circle. This result was proved by Carathéodory, and quoting Duren [2] it states:

Theorem 3.2 (Carathéodory Extension Theorem). *Let D by a domain bounded by a Jordan curve ∂D, and let f map D conformally onto the unit disk $\mathbb{D}$. Then f can be extended to a homeomorphism (i.e., f and f^{-1} are both continuous) of $\overline{D} = D \cup \partial D$ onto the closed disk $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$.*

3.1.2 Conformal invariance of Brownian motion

Let the sets $D \subset \mathbb{C}$ and $D' \subset \mathbb{C}$ be simply connected domains. Let $f : D \to D'$ be a conformal transformation, and let B_t be a Brownian motion with $B_0 = x \in D$, and $\tau_D = \inf\{t : B_t \notin D\}$. The following theorem [27] introduced by Lévy shows that Brownian motion is conformally invariant.

Theorem 3.3 (Conformal invariance of Brownian motion). *If*

$$A_s = \int_0^s |f'(B_t)|^2 dt \quad \text{and} \quad \sigma_t = \inf\{s : A_s \geq t\},$$

then $Y_t := f(B_{\sigma t})$ is a Brownian motion in D' started at $Y_0 = f(x)$ and stopped at $\tau_{D'} = \inf\{t : B_t \notin D'\}$.

3.2 Loewner's equation

The Loewner equation was first introduced in 1923 by Charles Loewner in order to prove a special case of the Bieberbach conjecture. The entire conjecture was eventually proved in 1985 by de Brnages using the the Loewner equation. In 1999 Oded Schramm introduced the stochastic Loewner evolution (SLE) while considering scaling limits of certain stochastic processes. It is now often referred to as the Schramm-Loewner evolution.

There are three related Loewner equations: 1) the radial equation, which is used for a cluster growing from a boundary to an interior point, 2) the whole plane equation, which is used for a cluster growing from one point to infinity, and 3) the chordal equation (for more details, see Lawler, Schramm and Werner [18]) which is used for clusters growing from a boundary point towards to a boundary point. The third type of Loewner equation is the one that we will be focusing on in this report. The chordal Loewner equation describes the time development of an analytic map into the upper half of the complex plane $\mathbb{C}$, obligating a defined singularity which moves around the real axis. Then, the applications of Loewner's equation use the outlines or traces of singularities in the upper half plane. Now, we will introduce Loewner's equation based on Lawler [15] or Rohde and Schramm [22].

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane of $\mathbb{C}$, and $h : \mathbb{H} \to \mathbb{H}$ be onto with $h(\infty) = \infty$, where h must be of the form $h(z) = az + b$, $a > 0$ and $b \in \mathbb{R}$. Now let $\gamma : [0, \infty) \to \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$ be a simple curve with

- $\gamma(0) = 0$,
- $\gamma(0, \infty) \subseteq \mathbb{H}$,
- $\gamma(t) \to \infty$ as $t \to \infty$.

For each $t \geq 0$ suppose that K is a bounded, compact set such that $\mathbb{H} \setminus K$ is simply connected, and $K_t := \gamma[0, t]$. Let $\mathbb{H}_t := \mathbb{H} \setminus K_t$ be the slit half plane which is simply connected by assumption. Thus, using the Riemann Mapping Theorem (Theorem 3.1) there exists a conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$ with g_t onto and $g_t(\infty) = \infty$.

Using the Schwartz reflection principle, as $z \to \infty$ we will expand the function g_t around ∞.

$$\therefore g_t(z) = bz + a_0 + \frac{a_1}{z} + O(\frac{1}{z^2}) \tag{3.4}$$

where $b > 0$ and $a_i \in \mathbb{R}$.

Consider the expansion of $f(z) = [g_t(1/z)]^{-1}$ about the origin. f locally maps $\mathbb{R}$ to $\mathbb{R}$ so that the coefficients in the expansion are real numbers and $b > 0$. For simplicity, the function g_t that should be used will be the unique g_t that satisfies the "hydrodynamic normalization" choosing $b = 1$ and $a_0 = 0$; that is,

$$\lim_{z\to\infty} (g_t(z) - z) = 0. \tag{3.5}$$

The constant $a_t := a_{K_t}$ only depends on the set K_t. Therefore, $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ with $g_t(\infty) = \infty$ is:

$$g_t(z) = z + \frac{a_1}{z} + O(\frac{1}{z^2}) \tag{3.6}$$

where $a_t := a_{K_t} = a(\gamma[0,\, t])$ is called the half-plane capacity from ∞.

Some relevant facts about a_t include:

1. $a_t = \lim_{z\to\infty} z(g_t(z) - z)$,

2. if $s < t$, then $a_s < a_t$,

3. $s \mapsto a_s$ is continuous,

4. $a_0 = 0$, $a_t \to \infty$ as $t \to \infty$.

Since it is possible to re-parameterize $\gamma(t)$ so that $a_t = 2t$, we assume $\gamma(t)$ under that condition. Now, suppose $K_t := \gamma[0, t]$ with $\mathbb{H}_t := \mathbb{H} \setminus K_t$ and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann maps. We let the image of $\gamma(t)$ be $U_t := g_t(\gamma(t))$ so that g_t satisfies the Loewner differential equation with the identity map $g_0 : \mathbb{H} \to \mathbb{H}$ as initial data.

Theorem 3.4 (Loewner 1923). *For fixed z, $g_t(z)$ is the solution of the IVP*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{3.7}$$

There are three different approaches on how to get the solution to equation (3.7):

1. Loewner's approach was to start with $\gamma(t)$, then find the function $g_t(z)$ and finally find the partial differential equation for $g_t(z)$

2. An alternative approach is to start with the partial differential equation for a given U_t, then solve for $g_t(z)$ and finally find the curve $\gamma(t); = g_t^{-1}(U_t)$.

3. This was the approach proposed by Schramm who suggested to start by letting $U_t = \sqrt{\kappa} B_t$ where B_t is the standard one dimensional Brownian motion so that $U_t \sim N(0, \kappa t)$, then solve for the function $g_t(z)$ and finally, find the curve $\gamma(t) := g_t^{-1}(U_t)$.

Based on Schramm's approach, suppose that the function $t \mapsto U_t$, $t \in [0, \infty)$ is continuous and real-valued, and then solving the Loewner equation gives the function g_t. Ideally we would like $g_t^{-1}(U_t)$ to be a well defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$. And although for most of the choices of U this is not possible, the next theorem gives a sufficient condition.

Theorem 3.5 (Rohde-Marshall 2001). *If U is Hölder $1/2$ continuous with sufficiently small Hölder $1/2$ norm, then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0, t]$.*

Moreover, a real-valued function U on $\mathbb{R}^d$ satisfies the Hölder condition if there are non-negative real constants C and α such that for every $x, y \in \mathbb{R}^d$,

$$|U(x) - U(y)| \leq C|x - y|^{\alpha} \tag{3.8}$$

where α is called the Hölder exponent.

3.3 Schramm-Loewner evolution (SLE)

As mentioned previously in this chapter, SLE has been and still is the best conjecture obtained for the scaling limit of a self-avoiding walk in dimension $d = 2$.

Therefore, we will proceed to introduce some ideas related to this process as well as the most important definitions and theorems as presented by Rohde and Schramm [22]. The suggestion of Schramm's is to let U_t be a Brownian motion, since we know it is conformally invariant, has independent and identically distributed increments, and is symmetric about the origin. Then, SLE with parameter κ (SLE_κ) is obtain by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one-dimensional Brownian motion with variance parameter κ.

Definition 3.6. SLE_κ in the upper half plane is the random collection of conformal maps $g_t(z)$ obtained by solving the chordal Loewner equation with $U_t = \sqrt{\kappa} B_t$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{3.9}$$

where B_t is a standard one dimensional Brownian motion with $\sqrt{\kappa} B_0 = x$.

Theorem 3.7 (Rohde-Schramm 2001). *There exist a curve $\gamma(t)$ associated to the SLE_κ.*

We see that Theorem 3.7 defines the family of conformally invariant measures SLE_κ on curves in the space $\mathbb{H}$. In addition, we observe that the SLE for certain values of the parameter κ corresponds to scaling limits of discrete lattice models. In particular:

- $\kappa = 6$ is the model for the boundaries of percolation clusters,
- $\kappa = 3$ is the model for the boundaries of Ising spin clusters,

- $\kappa = 4$ is the model for the harmonic explorer, and level lines of the Gaussian free field,
- $\kappa = 2$ is the model for the scaling limit of loop-erased random walk,
- $\kappa = \frac{8}{3}$ is the conjectured model for the scaling limit of a self-avoiding walk.

The results for SLE_κ with $\kappa = 6, 3, 4$ and 2 has been already proven, while the SLE with parameter $\kappa = 8/3$ still remains a conjecture for the scaling limit of the self-avoiding walk (SAW), which will be further analyzed in the next section.

Theorem 3.8 (Properties of SLE). *With probability one:*

- *for all* $0 < \kappa \leq 4$, $\gamma(t)$ *is a random, simple curve avoiding* $\mathbb{R}$,
- *for all* $4 < \kappa < 8$, $\gamma(t)$ *is a continuous but not a simple curve. It has double points, but does not cross itself.*
- *For all* $\kappa \geq 8$, $\gamma(t)$ *is a space filling curve. It has double points, but does not cross itself.*

For more information and a more detailed review about the basic properties of the Schramm-Loewner evolution (SLE), we refer the reader to Lawler [15] or Rohde and Schramm [22].

3.4 Conjectured scaling limit of SAW

For this section, we will assume that the discrete measure (uniform measure) used on the self-avoiding walks has a conformally invariant limit. Moreover, we will highlight the next property of the discrete measure of a SAW in the scaling limit given by Werner [27] assuming it exists and is conformally invariant:

Property 3.9. Given $\gamma[0,\, t]$, the conditional law or distribution of $\gamma[t,\, \infty)$ is identical to the law of $f^{-1}(\tilde{\gamma})$, where $\tilde{\gamma}$ is an independent copy of γ and f is a conformal map from $\mathbb{H}\backslash\gamma[0,\, t]$ onto $\mathbb{H}$ such that $f(\gamma_t) = 0$ and $f(\infty) = \infty$.

Therefore, the suggested curve γ that we need to find is a random continuous curve satisfying Property 3.9, as well as the condition that the curve should be symmetric with respect to the imaginary axis, in order for the laws of the image of γ and of γ to be identical.

Following equations (3.7) and (3.9), we recall that is possible to get the curve γ from the function U_t. Property 3.9 indicates that U_t is a stochastic process with independent increments, and as mentioned, it should be symmetric as well. Thus, the continuous function U_t must be a standard Brownian motion of the form $B_t = U_t/\sqrt{\kappa}$ with a variance $\kappa \geq 0$.

Summarizing, if the scaling limit of a self-avoiding walk exists and is conformally invariant, it can be obtained (for some given constant κ) by

$$\gamma(t) := g_t^{-1}(U_t) \tag{3.10}$$

if g_t^{-1} extends continuously to U_t.

Recall that the curve γ is the solution to equation (3.9) and therefore is called the chordal Schramm Loewner evolution (SLE_κ) with parameter κ.

Now, we will introduce some last ideas to complement the conformal restrictions that the conformally invariant scaling limit of a self-avoiding walk should satisfy. Also we will complement this with the next theorems about the chordal restriction property given by Werner [27] and Lawler, Schramm and Werner [18].

Theorem 3.10. *There exists a unique probability measure supported on continuous paths without double points that satisfies conformal restriction. It is the chordal Schramm-Loewner Evolution (SLE) with parameter 8/3, and it is supported on curves with fractal dimension 4/3.*

Theorem 3.11 (Lawler, Schramm, Werner). *For $\kappa = 8/3$, chordal SLE in the half plane satisfies:*

$$P(\gamma \cap K_t = \emptyset) = \Phi'_{K_t}(0)^{5/8} \tag{3.11}$$

where the conformal transformation $\Phi_{K_t} : \mathbb{H} \setminus K_t \to \mathbb{H}$ sends $0 \mapsto 0$ and $\infty \mapsto \infty$ with $\Phi'_{K_t}(\infty) = 1$.

Fix an H, so that for some exponent α, $P(\gamma \subset H) = \Phi'_H(0)^\alpha$, recalling Property 3.9 the conditional distribution of $g_t(\gamma[t, \infty)) - U_t$ is the same as the distribution of γ.

In particular for any H',

$$P[g_t(\gamma[t, \infty)) - U_t \subset H' | \gamma[0, t]] = \Phi'_{H'}(0)^\alpha. \tag{3.12}$$

Since $\gamma[t, \infty) \subset H$ implies $g_t(\gamma[t, \infty)) \subset g_t(H)$. Then,

$$P[\gamma \subset H | \gamma[0, t]] = \Phi'_{g_t(H)-U_t}(0)^\alpha = \Phi'_{g_t(H)}(U_t)^\alpha. \tag{3.13}$$

Equation (3.13) holds only for one specific choice of the variable κ, namely $\kappa = 8/3$. Since the limit should satisfy conformal restriction, the conjectured scaling limit for a self-avoiding walk (SAW) in dimension $d = 2$ is $\text{SLE}_{8/3}$; in other words, the Schramm-Loewner evolution with parameter $\kappa = 8/3$. To be more formal, we quote Lawler [15].

Conjecture 3.12. *The continuum scaling limit of the measure on self-avoiding walk is* $SLE_{8/3}$.

In the next chapter we outline some of the work done by Kennedy to provide numerical evidence to strongly support this conjecture.

Chapter 4

Tests of SLE predictions for the 2D SAW

As we pointed out in the last chapter, the most recent investigations and research about self-avoiding walks in dimension $d = 2$ have only gone so far as proposing conjectures about the scaling limit of the SAW in the half plane, this being the stochastic (or Schramm-) Loewner evolution with parameter $\kappa = 8/3$ ($\mathrm{SLE}_{8/3}$). Currently, Tom Kennedy [7], [8], [9], [10] and [11] has carried out some Monte Carlo simulation tests of this prediction that the scaling limit of the self-avoiding walk is $\mathrm{SLE}_{8/3}$ and he discovered that there is an excellent fit or adjustment between these two distributions, therefore providing numerical support for the conjecture that if the limit exists and is conformally invariant, then it must be the Schramm-Loewner evolution (SLE) with $\kappa = 8/3$.

For the purpose of testing the agreement between $\mathrm{SLE}_{8/3}$ and self-avoiding walk, Kennedy computed the probability distributions of certain functionals of $\mathrm{SLE}_{8/3}$ and compared these to simulations of the corresponding fuctionals for the self-avoiding walk, obtaining the result that they present a good test for the mentioned conjecture. It is important to remark that the distribution of these functionals for $\text{SLE}_{8/3}$ was obtained using Theorem 3.11, while the simulation of the corresponding functionals for the self-avoiding walk was done by implementing the pivot algorithm (see Madras and Slade [20] for further details).

We present the following example to demonstrate how can we apply equations (3.11), (3.12) and (3.13) from Theorem 3.11 to obtain the distribution of certain random variables when we have an explicit formula for the conformal transformation from $\mathbb{H} \setminus A$ onto $\mathbb{H}$.

Example 4.1 (Kennedy [8]). Fix a point $(c, 0)$ on the real axis. Given an $\text{SLE}_{8/3}$ path, consider the distance from the curve to $(c, 0)$. Let

$$X = \frac{1}{c} \inf_{t \geq 0} |\gamma(t) - (c, 0)|$$

be the ratio of the distance to c. Note that X takes values in $(0, 1]$. Since SLE is invariant under dilations, the distribution of X is independent of c. Take $c = 1$, and for $a < 1$, let $A_a = \{z \in \mathbb{C} : |z - 1| < a\} \cap \mathbb{H}$. Then, the distance X from $\gamma[0, \infty)$ to $(1, 0)$ is less than or equal to a if and only if $\gamma[0, \infty)$ hits A_a. Thus, if $z = 0$, then from Theorem 3.11 it follows that

$$P\{X \leq a\} = P\{\gamma[0, \infty) \cap A_a \neq \emptyset\} = 1 - \Phi'_{A_a}(z)^{5/8}$$

where Φ_{A_a} is given by

$$\Phi_{A_a}(z) = z - 1 + \frac{a^2}{z - 1} + 1 + a^2$$

and is the conformal transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ normalized to send $0 \mapsto 0$ and $\infty \mapsto \infty$ with $\Phi'_{A_a}(\infty) = 1$. Since $\Phi'_{A_a}(0) = 1 - a^2$, we apply Theorem 3.11 to conclude

$$P\{X \le t\} = 1 - (1 - t^2)^{5/8}.$$

For the next step after simulating the distributions of the SAW and the $\text{SLE}_{8/3}$ curves, Kennedy plotted the difference of these distributions finding that the fit between them was excellent. Quoting Kennedy [8], "simulations of the SAW in a half plane have shown that the distribution of two particular random variables related to the walk agree extremely well with the exact distribution of $\text{SLE}_{8/3}$ for these random variables. This supports the conjecture that the scaling limit of the SAW is $\text{SLE}_{8/3}$".

We should remark that the scaling limit must be constructed as follows. For a domain D and two points z and w on its boundary, we introduce a lattice $D_\delta = \delta\mathbb{Z}^2 \cap D$, where $\delta > 0$ is the lattice spacing, and consider all self-avoiding walks which start at z and end at w. The probability of this walk should be taken to be proportional to ξ^{-N} where N is the number of steps in the walk, and ξ is the constant such that the number of self-avoiding walks in the plane starting at the origin grows with the number of steps N as ξ^{-N}.

The measure must be normalized so that it will be a probability measure. Then we take the limit of this measure when $\delta \to 0$.

Other work by Kennedy [9] confirms that when testing the conjecture made by Lawler, Schramm and Werner that the scaling limit of the self-avoiding walk for two dimensions is provided by the Schramm-Loewner evolution with $\kappa = 8/3$. In addition this work also found excellent results when testing the conformal invariance of the scaling limit of the SAW.

Moreover, Kennedy has presented some work done on how to re-parameterize either the natural parameterization of the scaling limit of the self-avoiding walk or the standard parameterization of the SLE in the half plane to make them correspond with each other, i.e., agree as parameterized curves.

Kennedy argues [11] that after doing Monte Carlo simulations and finding good agreement between curves, what he called p-variation of the Schramm-Loewner evolution (SLE) curve, gives a parameterization that corresponds to the one of the self-avoiding walk (SAW).

Therefore, quoting Kennedy "SLE with p-variation as its parameterization and SAW with its natural parameterization should agree as parameterized curves". Currently efforts to establish this rigorously are underway.

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