SCALING LIMIT OF LOOP-ERASED RANDOM WALK

A REPORT

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND STATISTICS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

IN STATISTICS

UNIVERSITY OF REGINA

By

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Regina, Saskatchewan

April 2007

Abstract

The loop-erased random walk (LERW) was first studied in 1980 by Lawler as an attempt to analyze self-avoiding walk (SAW) which provides a model for the growth of a linear polymer in a good solvent. The self-avoiding walk is simply a path on a lattice that does not visit the same site more than once. Proving things about the collection of all such paths is a formidable challenge to rigorous mathematical methods. Eventually, it was discovered that SAW and LERW are in different universality classes. LERW is a model for a random simple path with important applications in combinatorics, computer science and quantum field theory. This model has continued to receive attention in recent years, in part because of connections with uniform spanning trees.

This report contains some of the most important results about loop-erased random walk and its scaling limit in several dimensions. Although there is an extensive body of work concerning LERW, we will rather give a summary of the key properties and results. The first two chapter of this report provide the preliminaries for the loop-erased random walk. In Chapter 1, the necessary background and history about loop-erased random walk and its scaling limit is presented. Chapter 2 introduces

some aspects and definition of LERW for $d \geq 3$. In Chapter 3, it is shown that LERW converges to Brownian motion for $d \geq 5$. In Chapter 4, the same result as in Chapter 3 holds. It is shown that LERW for $d = 4$ converges to Brownian motion, but a logarithmic correction to the scale is needed. A promising paper by Kozma is introduced in Chapter 5 which shows that the scaling limit of LERW in $d = 3$ exists and is invariant under rotations and dilatations. Chapter 6 presents some important concepts and facts from complex analysis including the Loewner equation, and gives an introduction to the stochastic Loewner evolution (SLE). At the end of this chapter, it is shown that the scaling limit of LERW for $d = 2$ is equal to the *radial* SLE_2 path. The final chapter reviews Wilson's algorithm which generates uniform spanning trees (UST) using LERW.

Acknowledgements

I would not have been able to complete this journey without the aid and support of countless people over the past two years.

I would like to thank my supervisor Dr. Michael Kozdron, whose expertise, understanding and patience added considerably to my graduate experience. I have greatly appreciated his guidance and support during my graduate career and throughout the completion of this report. In addition, I wish to thank the faculty and staff of the Department of Mathematics and Statistics at the University of Regina, whose efforts have made the department a wonderful place to study in the past two years.

I would also like to express my gratitude to my uncle and aunt, my grandparents and all of my friends for their help and support when I needed it most.

I cannot end without thanking the most important people in my life, my parents Carlos and Yolanda, my brothers Jorge and Carlos, my sister Gabriela and my girlfriend Shanna and her family, on whose constant encouragement, support and love I have relied on throughout my time in Canada.

Post-Defense Acknowledgements

I would like to express my appreciation to Dr. Christian Beneš for his expert advice, useful comments and careful reading of this report. My gratitude to Dr. Douglas Farenick for his support and excellent job as chair of my defense and as coordinator of graduate students for the Department of Mathematics and Statistics at the University of Regina. Finally, I would like to thank Dr. Andrei Volodin for his help during these two years, and for his precious time and comments on this report as a member of my examining committee.

Dedication

To the memory of my grandfather Carlos

Contents

Chapter 1

Introduction

As written by Schramm [22], it is commonly the case when discrete probabilistic models are considered as a substitute, or simplification, of a continuous process. There are definite advantages for working in the discrete setting, where unpleasant technicalities and difficulties can frequently be avoided, the setup is easier to understand, and simulations are possible. On the other hand, there are numerous cases where the continuous case is easier to analyze than the discrete model. In these situations the continuous model may be a useful simplification of the discrete case.

The relation between both discrete and continuous processes is of fundamental importance in probability theory and its applications. One way to define a continuous process is by taking a scaling limit of a discrete process. This means making sense of the limit of a sequence of grid processes on finer and finer grids.

The simplest and most important example of the connection between grid-based models and continuous process is the scaling limit of the simple random walk (SRW) to define Brownian motion, which is well studied and quite well understood. Simple random walk on any integer lattice in $\mathbb{R}^d$ converges to d-dimensional Brownian motion in the scaling limit.

There are a number of interesting models associated to simple random walk in which each step depends on the past in a complicated manner. All are more difficult to analyze than the usual simple random walk. Two of these models are the self-avoiding walk (SAW) [20] and the loop-erased random walk (LERW) [15].

The study of self-avoiding walks (SAW) originated in chemical physics as an attempt to model polymer chains. A polymer consists of many monomers which form a random chain with the constraint that different monomers cannot occupy the same space. The self-avoiding walk is defined as the uniform measure on random walk paths of a given length conditioned on no self-intersection. On a rigorous level, this mathematical model itself has proven to be an extremely difficult one to analyze. Lawler in 1980 [12] introduced the loop-erased random walk with the hope of getting some perspective on the problem of the self-avoiding walk. It turned out that the two processes are in different universality classes.

Nonetheless, loop-erased random walk (LERW) is an interesting model in its own right with important applications in combinatorics, physics and quantum field theory.

Before proceeding to analyze the LERW and its scaling limit in several dimensions, we will give some essential definitions and facts that will help to comprehend the content of this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, which means that Ω is a space, $\mathcal{F}$ is a σ-algebra of subsets of Ω, $\mathbb{P}$ is a countably additive, non-negative measure on $(\Omega, \mathcal{F})$ with total mass $P(\Omega) = 1$. A *discrete time stochastic process* $S = \{S_n, n = 0, 1, 2, \ldots\}$ is defined as a countable collection of random variables indexed by non-negative integers.

Let $\mathbb{Z}^d = \{(z_1, \ldots, z_d) : z_1 \in \mathbb{Z}, \ldots, z_d \in \mathbb{Z}\} \subseteq \mathbb{R}^d$ be the d-dimensional integer lattice. Let $X_1, X_2 \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in the integer lattice $\mathbb{Z}^d$ with

$$P\{X_j = e\} = \frac{1}{2d}, \qquad |e| = 1.$$

A *simple random walk* starting at $x \in \mathbb{Z}^d$ is a discrete time stochastic process $S = \{S_n,\, n = 0, 1, 2, \ldots\}$, with $S_0 = x$ and

$$S_n = x + X_1 + \cdots + X_n.$$

The probability distribution of S_n is denoted by

$$p_n(x, y) = P^x\{S_n = y\}.$$

A *continuous time stochastic process* $B = \{B_t,\, 0 \le t < \infty\}$ is an uncountable collection of random variables indexed by non-negative real numbers.

Consider a continuous time stochastic process $B = \{B_t,\, 0 \le t < \infty\}$ having the following properties:

- $B_0 = 0$,
- for $0 \le s \le t < \infty$, $B_t - B_s \sim N(0, t - s)$,

- for $0 \leq s \leq t < \infty$, $B_t - B_s$ is independent of B_s,

- the trajectories $t \mapsto B_t$ are continuous.

The process $B = \{B_t, 0 \leq t < \infty\}$ is called (one-dimensional, standard) *Brownian motion*. This mathematical object is used to describe many phenomena such as the random movements of minute particles, stock market fluctuations, and the evolution of physical characteristics in the fossil record. If we let $B = (B^1, \ldots, B^d)$ where the components B^j are independent, one-dimensional Brownian motions, then we say that B is a d-dimensional Brownian motion.

Lawler [15] defines *loop-erased random walk* on the integer lattice as a process obtained by erasing the loops of a simple random walk chronologically. The result of this process has self-avoiding paths.

The appropriate way to discuss limiting behaviour is through convergence. The concept of convergence is essential in probability theory, and its applications to statistics and stochastic processes.

Among all types of convergence, this paper will deal primarily with the convergence of probability measures.

Jacod and Protter [6] define *weak convergence* and *convergence in distribution* (which are closely related) as follows.

Let μ_n and μ be probability measures on $\mathbb{R}^d$ $(d \geq 1)$. The sequence μ_n converges weakly to to μ if $\int f d\mu_n$ converges to $\int f d\mu$ for each f which is real-valued, continuous and bounded on $\mathbb{R}^d$, i.e.,

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu.$$

Furthermore, let $X_1, \ldots, X_n$ be $\mathbb{R}^d$-valued random variables. We say X_n converges in distribution to X if the distribution measures P^{X_n} converge weakly to P^X where P^{X_n} is defined by $P^{X_n}(A) = P(X_n \in A)$ for A Borel and P^X is defined similarly. We write $X_n \xrightarrow{D} X$.

Note that in this type of convergence, it is the probability distribution that converge, and not the values of the random variables themselves.

Theorem 1.1 (Portmanteau theorem). *If μ_n, μ are probability measures on $(\mathbb{R}, \mathcal{B})$, then the following are equivalent:*

- $\mu_n \to \mu$ *in distribution as* $n \to \infty$,
- $\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu \quad \forall\ f \in C_b$ *(bounded,continuous),*
- $\limsup_{n\to\infty} \mu_n(F) \le \mu(F) \quad \forall$ *closed* $F \subseteq \mathbb{R}$,
- $\liminf_{n\to\infty} \mu_n(E) \ge \mu(E) \quad \forall$ *open* $E \subseteq \mathbb{R}$,
- $\lim_{n\to\infty} \mu_n(A) = \mu(A) \quad \forall$ *Borel* A *with* $\mu(\partial A) = 0$.

Example 1.2. Let $\Omega = [0, 1]$, let $\mathcal{F}$ be the Borel sets of Ω, let $A \in \mathcal{F}$, let $x_n = \frac{1}{n}$ and define the following probability measures:

$$\mu_n(A) = \begin{cases} 1, & \text{if } x_n \in A, \\ 0, & \text{if } x_n \notin A, \end{cases} \qquad \mu(A) = \begin{cases} 1, & \text{if } 0 \in A, \\ 0, & \text{if } 0 \notin A. \end{cases}$$

Now let f be continuous and bounded, so that

$$\int_{-\infty}^{\infty} f(x)d\mu_n(x) = f(x_n) = f(1/n) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)d\mu(x) = f(0).$$

Clearly $x_n \to 0$ as $n \to \infty$, and so since f is continuous, we find

$$f(x_n) \to f(0) \qquad as \;\; n \to \infty,$$

which shows that $\mu_n \to \mu$ weakly. We now show that strict inequalities hold in the second and third parts of the Portmanteau theorem. Let $A = (0, 1)$ which is an open set. We see that $\mu(A) = 0$ since $0 \notin A$ and that $\mu_n(A) = 1 \; \forall \, n$ since $x_n \in A$. Therefore,

$$\liminf_{n\to\infty} \mu_n(A) = 1 > \mu(A) = 0.$$

Let $A = \{0\}$ which is a closed set. We see that $\mu(A) = 1$ since $0 \in A$ and $\mu_n(A) = 0$ $\forall \, n$ since $x_n \notin A$. Therefore,

$$\limsup_{n\to\infty} \mu_n(A) = 0 < \mu(A) = 1.$$

The scaling limit for the LERW in dimension 1 is trivial. If you take a random walk with very small steps you get an approximation to Brownian motion. To be more precise, if the step size is ξ, one needs to take a walk of length L/ξ^2 to approximate a Brownian motion of length L. As the step size tends to 0 (and the number of steps increase comparatively), the LERW converges to Brownian motion in distribution.

In dimension 2, the LERW was conjectured to be conformally invariant in the limit as the lattice becomes finer and finer, which allowed physicist to make conjectures about fractal dimensions and critical exponents. There are three different approaches to prove the scaling limit of LERW in dimension 2: the connection to random domino tilings [7, 8], the connection to the stochastic Loewner evolution (SLE) [17], and a naïve approach proposed by Kozma [10].

Attempts to understand LERW in dimension 3 are focused mainly on the number of steps it takes to reach the distance r. Physicists argue that this value is $\approx r^{\xi}$ and did numerical experiments to show that $\xi = 1.62 \pm 0.01$ [5]. Strictly, the existence of ξ has not been proved, therefore we must talk about upper and lower exponents ($\underline{\xi} \leq \bar{\xi}$). The best estimates known are $1 \leq \underline{\xi} \leq \bar{\xi} \leq 5/3$ [15]. It is unclear whether LERW has a natural continuum equivalent in dimensions smaller than 4. Note that Brownian motion has a dense set of loops and therefore it is not clear how to delete them in chronological order.

The dimension 4 is critical for LERW. A weak law for the remaining points after erasing loops has been proved. From this, it has been shown that the process approaches Brownian motion with a logarithmic correction to the scaling [13]. This has been proved with no use of the difficult technique of lace expansion [23] (which is used to obtain high dimensional results about the self-avoiding walk).

Above dimension 4, the number of points remaining after erasing loops is a positive fraction of the total number of the points. It turns out that if one takes a random walk of length n, its loop-erasure has length of the same order of magnitude. Scaling accordingly, it results that as n goes to infinity, the LERW converges in distribution to Brownian motion. Lawler [13] proved it using the non-intersection of simple random walks.

In the remaining chapters, we will discuss the scaling limit of LERW in each of the cases $d = 2$, $d = 3$, $d = 4$, and $d \geq 5$ separately.

Chapter 2

LERW in $d \geq 3$

We will start this chapter by defining the loop-erasing procedure which assigns to each finite simple random walk path λ a self-avoiding path $L\lambda$ [13]. Let $\lambda = [\lambda(0), \ldots, \lambda(m)]$ be a simple random walk path of length m. If λ is already self-avoiding set $L\lambda = \lambda$. If not, let

$$s_0 = \sup\{j : \lambda(j) = \lambda(0)\},$$

and for $i > 0$, let

$$s_i = \sup\{j : \lambda(j) = \lambda(s_{i-1} + 1)\}.$$

If

$$n = \inf\{i : s_i = m\},$$

then

$$L\lambda = [\lambda(s_0), \lambda(s_1), \ldots, \lambda(s_n)].$$

Note that the loop-erasing procedure "erases" loops in chronological order. If we want to erase the loops in the reverse direction, let

$$\lambda_R(j) = \lambda(m-j), \qquad 0 \le j \le m,$$

and define the reverse loop-erasing operator L^R by

$$L^R(\lambda) = (L\lambda_R)_R.$$

It is not difficult to construct λ such that $L\lambda \neq L^R\lambda$. However, it can be proved that $L\lambda$ and $L^R\lambda$ have the same distribution if λ is chosen using the distribution of simple random walk. See Lemma 7.2.1 of [13].

Now that the loop-erasure process has been explained, we are ready to define the loop-erased random walk of an infinite simple random walk for $d \ge 3$.

Suppose that S_j is a simple random walk in $\mathbb{Z}^d$, $d \ge 3$, and let

$$\sigma_0 = \sup\{j : S_j = 0\},$$

and for $i > 0$,

$$\sigma_i = \sup\{j > \sigma_{i-1} : S_j = S(\sigma_{i-1}+1)\}.$$

Then, let

$$\hat{S}(i) = S(\sigma_i).$$

We call $\hat{S}$ *the loop-erased self avoiding random walk* or, more compactly, *the loop-erased random walk*. Notice that $\hat{S}$ is well defined since simple random walk is transient for $d \ge 3$. Recall this means that simple random walk is not expected to visit 0 infinitely often, i.e.,

$$P\{S_n = 0 \text{ infintely often}\} = 0.$$

In order to analyze the behaviour of $\hat{S}(n)$ for large n. It is important to investigate how many steps of the simple random walk remain after erasing the loops. We will consider the case $d \geq 3$ where the loop-erased random walk is constructed by erasing loops from an infinite simple random walk. An equivalent way of describing σ_i is as follows

$$\sigma(i) = \sigma_i = \sup\{j : S(j) = \hat{S}(i)\}.$$

Now, let $\rho(j)$ be the "inverse" of $\sigma(i)$. That is, define

$$\rho(j) = i \ \text{ if } \sigma_i \leq j < \sigma_{i+1}.$$

Then,

$$\rho(\sigma(i)) = i, \tag{2.1}$$

$$\sigma(\rho(j)) \leq j,$$

and

$$\hat{S}(i) = S(\sigma(i)).$$

If Y_n is the indicator function of the event "the n^{th} point of S is not erased", then

$$Y_n = \begin{cases} 1, & \text{if } \sigma_i = n \text{ for some } i \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and so

$$\rho(n) = \sum_{j=0}^{n} Y_j$$

where $\rho(n)$ is the number of points remaining of the first n points after loops are erased. Now, let $a_n = E(Y_n)$ be the probability that the n^{th} point is not erased. Let

w_n be the path resulting after erasing loops on the first n steps of S, i.e.,

$$w_n = L[S_0, \ldots, S_n].$$

Then $Y_n = 1$ if and only if the loop-erased random walk up to time n and the simple random walk after time n do not intersect. More precisely, $Y_n = 1$ iff

$$w_n \cap S[n+1, \infty) = \emptyset,$$

Therefore,

$$P\{Y_n = 1 | S_0, \ldots, S_n\} = E_{S_{w_n}}(S_n)$$

where $E_{S_{w_n}}$ is the expectation assuming the simple random walk starts after the loop-erased random walk up to time n, and so

$$a_n = E[E_{S_{w_n}}(S_n)].$$

This can be restarted by translating the origin. If S^1, S^2 are independent simple random walks starting at the origin, then

$$a_n = P\{L^R[S_0^1, \ldots, S_n^1] \cap S^2[1, \infty) = \emptyset\},$$

and since $L^R\lambda$ and $L\lambda$ have the same distribution for a simple random walk, we get

$$a_n = P\{L[S_0^1, \ldots, S_n^1] \cap S^2[1, \infty) = \emptyset\}.$$

The goal is to derive an upper bound which essentially states that $\rho(n)$ grows no faster than $n(\ln n)^{-1/3}$ for $d = 4$. Define

$$b_n = E(\rho(n)) = \sum_{j=0}^{n} E(Y_j).$$

Theorem 2.1 (Lawler). *If* $d = 4$,

$$\limsup_{n\to\infty} \frac{\ln b_n - \ln n}{\ln \ln n} \le -\frac{1}{3}.$$

Remark. *In the proof of the theorem a lower bound on the probability of returning to the origin while avoiding a given set is needed. However we will not give the proof of this theorem in this paper and instead refer the reader to Chapter 7 of [13].*

It is natural to ask how good the bound in this theorem is. We will consider the case $d = 3, 4, 5$. Then

$$Z_n = E_{S_{w_n}}(S_n),$$

$a_n = E(Z_n)$ and $\rho_n = \sum_{j=0}^{n} a_j$. Lawler in [15] gives a way to estimate $E(Z_n)$ by $E(Z_n^3)^{1/3}$. While this only gives a bound for this quantity in one direction, in the case of $d = 4$, using the ideas of slowly recurrent sets for simple random walk [14], it has been shown that this bound is sharp and that

$$E(Z_n^3) \approx \begin{cases} 1, & d \ge 5, \\ (\ln n)^{-1}, & d = 4, \\ n^{-1/2}, & d = 3. \end{cases}$$

It is quite likely that this bound is not sharp in dimensions below four.

An important question is how much do we lose when we estimate $E(Z_n)$ by $E(Z_n^3)^{1/3}$? If $d = 4$, we lose very little:

$$a_n \approx (\ln n)^{-1/3}, \;\; d = 4.$$

For $d = 3$, we expect that the estimate for $E(Z_n)$ is not sharp and that $a_n \approx n^{-\alpha}$ for some $\alpha > \frac{1}{6}$.

Chapter 3

Convergence of LERW to Brownian motion in $d \geq 5$

In this chapter we will show that a loop-erased random walk with an appropriate scale converges to Brownian motion. For $d \geq 5$, the scaling will be a constant times the usual scaling for simple random walk. Further details about the results in this chapter may be found in Chapter 7 of [13].

To prove that LERW converges to Brownian motion, it is necessary to show that the loop-erasing process is uniform on paths, i.e.,

$$\frac{\rho(n)}{r_n} \to 1, \quad \text{for some } r_n \to \infty.$$

It is convenient to consider S as a two-sided walk. Let S^1 be a simple random walk independent of S and extend S_j, $-\infty < j < \infty$, by defining

$$S_j = \begin{cases} S_j, & 0 \leq j < \infty, \\ S^1_j, & -\infty < j \leq 0. \end{cases}$$

We call a time j loop-free for S if $S(-\infty, j] \cap S(j, \infty) = \emptyset$. Since the number of intersection probabilities for simple random walks for $d \geq 5$ is finite, then for each j

$$P\{j \text{ loop-free}\} = P\{S(-\infty, 0] \cap S(0, \infty) = \emptyset\} = b > 0.$$

We will mention a lemma and a theorem due to Lawler [13] that are fundamental to prove the main result. We believe that the proofs are instructive and so we include them in this report.

Lemma 3.1 (Lawler). *If $d \geq 5$, with probability one, $S(-\infty, \infty)$ has infinitely many positive loop-free points and infinitely many negative loop-free points.*

Proof. Let X be the number of positive loop-free points. A time j is n-loop-free if $S[j-n, j] \cap S(j, j+n] = \emptyset$. Then

$$P\{j \ n\text{-loop-free}\} = P\{S[-n, 0] \cap S(0, n] = \emptyset\} = b_n,$$

and $b_n \to b$. Now we let $V_{i,j}$ be the event $\{(2i-1)n \text{ is loop-free}\}$ and $W_{i,n}$ the event $\{(2i-1)n \text{ is } n\text{-loop-free}\}$. We can notice that for a given n, the events $W_{i,n}$, $i = 1, 2, \ldots$, are independent. For any $k < \infty$, $\epsilon > 0$, find m such that if Y is a binomial random variable with parameters m and ϵ, $P\{Y < k\} \leq \epsilon$, then

$$\begin{aligned} P\{X \geq k\} &\geq P\{\sum_{i=1}^{m} I(V_{i,n}) \geq k\} \\ &\geq P\{\sum_{i=1}^{m} I(W_{i,n}) \geq k\} - P\{\sum_{i=1}^{m} I(W_{i,n} \backslash V_{i,n}) \geq 1\} \\ &\geq 1 - \epsilon - m(b - b_n). \end{aligned}$$

Now if we choose n so that $m(b-b_n) \leq \epsilon$. Then $P\{X \geq k\} \geq 1-2\epsilon$. Since this holds for all $k < \infty$, $\epsilon > 0$, we have that $P\{X = \infty\} = 1$. A similar proof shows that the number of negative loop-free points is infinite with probability one. □

Theorem 3.2 (Lawler). *If $d \geq 5$, there exists an $a > 0$ such that with probability one*

$$\lim_{n\to\infty} \frac{\rho(n)}{n} = a.$$

Proof. First we have to order the loop-free points of $S(-\infty, \infty)$,

$$\cdots \leq j_{-2} \leq j_{-1} \leq j_0 \leq j_1 \leq j_2 \leq \cdots,$$

with

$$j_0 = \inf\{j \geq 0 : j \text{ loop-free}\}.$$

We can erase loops on the two-sided path $S(-\infty, \infty)$ by erasing separately on each piece $S[j_i, j_{i+1}]$. Let $\tilde{Y}_n$ be the indicator function of the event "the n^{th} point is not erased in this procedure", i.e., $\tilde{Y}_n = 1$ if and only if $j_i \leq n < j_{i+1}$ for some i and

$$L(S[j_i, n]) \cap S(n, j_{i+1}] = \emptyset.$$

We can see that the $\tilde{Y}_n$ form a stationary, ergodic sequence. Therefore, by a standard ergodic theorem (see [3], Theorem 6.28), with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \tilde{Y}_j = E(\tilde{Y}_0).$$

If instead we erase the loops only on the path $S[0, \infty)$, ignoring $S(-\infty, 0)$, the self-avoiding path we get may be sightly different. However, it is easy to see that if

$n \geq j_0$, then $Y_n = \tilde{Y}_n$, where Y_n is as defined in the first part of Chapter 2. Therefore, since $j_0 < \infty$, with probability one,

$$\lim_{n\to\infty} \frac{\rho(n)}{n} = \frac{1}{n}\sum_{j=0}^{\infty} Y_n = E(\tilde{Y}_0) = a.$$

To see that $a > 0$ we can note that

$$a \geq P\{0 \text{ loop-free}\} > 0. \qquad \square$$

We are now ready to proceed with the main theorem also due to Lawler [13].

Theorem 3.3 (Main Result). *If $d \geq 5$ and*

$$\hat{W}_n(t) = \frac{d\sqrt{a}\hat{S}([nt])}{\sqrt{n}},$$

then $\hat{W}_n(t)$ converges in distribution to $B(t)$, where B is a standard Brownian motion.

Proof. Recall that $\hat{S}(n) = S(\sigma(n))$, then by Theorem 3.2 with probability one,

$$\lim_{n\to\infty} \frac{\rho(\sigma(n))}{\sigma(n)} = a,$$

and hence by (2.1),

$$\lim_{n\to\infty} \frac{\sigma(n)}{n} = \frac{1}{a}. \tag{3.1}$$

The standard invariance principle (Donsker's theorem) states that if

$$W_n(t) = dn^{-1/2}S([nt]),$$

then

$$W_n(t) \to B(t),$$

in the metric space $C[0,1]$, where B is a standard Brownian motion in $\mathbb{R}^d$. Now suppose that $b_n \to \infty$ and

$$r_n(t) = \frac{\sigma([nt])}{b_n} \to t.$$

Then by the tightness in $C[0,1]$ of the sequence W_n,

$$\frac{S(\sigma([nt]))}{\sqrt{b_n}} - \frac{S([b_n t])}{\sqrt{b_n}} \to 0,$$

and hence

$$\frac{dS(\sigma([nt]))}{\sqrt{b_n}} \to B(t).$$

It follows from (3.1) that

$$\frac{\sigma(n)a}{n} \to 1,$$

and so

$$\frac{\sigma([nt])a}{n} \to t.$$

Since $r_n \to t$, we therefore have

$$\frac{\sigma([nt])a}{r_n(t)} \to 1,$$

which shows that the loop-erasing procedure is uniform on paths. $\square$

Chapter 4

Convergence of LERW to Brownian motion in $d = 4$

LERW has a critical dimension of four. In the critical dimension $d = 4$, we can prove a weak law for the number of points remaining. From this, it can be shown that the process converges to Brownian motion, although a logarithmic correction to the scaling is needed ([13], Chapter 7). Just like $d \geq 5$ the key step to proving such convergence is to show that the loop-erasing process is uniform on paths, i.e.,

$$r_n^{-1}\rho(n) \to 1, \quad \text{for some } r_n \to \infty.$$

To prove this we cannot use the same procedure as for $d \geq 5$ since $S(-\infty, \infty)$ contains no (two-sided) loop-free points. However, it will be possible to make use of one-sided loop-free points. Let $I_n = I(n)$ be the indicator function of the event "n is a (one-sided) loop-free point," i.e.,

$$S[0, n] \cap S(n, \infty) = \emptyset.$$

The next lemma due to Lawler [13] shows that the property of being loop-free is in some sense a local property.

Lemma 4.1 (Lawler). *Let* $d = 4$ *and*

$$U_n = \{S[0,n] \cap S(n,\infty) = \emptyset\},$$

$$V_{n,k} = \{S[k - n(\ln n)^{-9}, k] \cap S(k, k + n(\ln n)^{-9}] = \emptyset\}.$$

Then for all k *with* $n(\ln n)^{-9} \le k \le n$,

$$P(V_{n,k}) = P(U_n)(1 + O(\frac{\ln\ln n}{\ln n})).$$

Proof. It suffices to prove this lemma for $k = n$. We write U for U_n and V for $V_{n,n}$. Let

$$\bar{V} = \bar{V}_n = \{S[n - n(\ln n)^{-9}, n] \cap S(n,\infty) = \emptyset\}.$$

Then

$$P(\bar{V}) = P(U)(1 + O(\frac{\ln\ln n}{\ln n})). \tag{4.1}$$

Let

$$W = W_n = \{S[n - n(\ln n)^{-18}, n] \cap S(n, n + n(\ln n)^{-9}] = \emptyset\},$$

$$\bar{W} = \bar{W}_n = \{S[n - n(\ln n)^{-18}, n] \cap S(n,\infty) = \emptyset\}.$$

Then

$$P(\bar{W}) = P(\bar{V})(1 + O(\frac{\ln\ln n}{\ln n})). \tag{4.2}$$

But

$$P(W\backslash\bar{W}) \le P\{S[n - n(\ln n)^{-18}, n] \cap S[n + n(\ln n)^{-9}, \infty) = \emptyset\}$$
$$= o((\ln n)^{-2}).$$

Since $P(\bar{W}) \approx (\ln n)^{-1/2}$, this implies that

$$P(W) = P(\bar{W})(1 + O(\frac{\ln\ln n}{\ln n})). \tag{4.3}$$

But $\bar{V} \subset V \subset W$, so (4.1)–(4.3) imply that

$$P(V) = P(U)(1 + O(\frac{\ln\ln n}{\ln n})). \qquad \square$$

The next lemma will show that there are a lot of loop-free points on a path. Suppose $0 \le j < k < \infty$, and let $Z(j,k)$ be the indicator function of the event "there is no loop-free point between j and k", i.e.,

$$\{I_m = 0, j \le m \le k\}.$$

Then if $d = 4$,

$$E(Z(n - n(\ln n)^{-6}, n)) \ge P\{S[0, n - n(\ln n)^{-6}] \cap S(n+1, \infty) \ne \emptyset\}$$
$$\ge c\frac{\ln\ln n}{\ln n}.$$

The next lemma due to Lawler [13] gives a similar bound in the opposite direction.

Lemma 4.2 (Lawler). *If $d = 4$, for any n and k with $n(\ln n)^{-6} \le k \le n$,*

$$E(Z(k - n(\ln n)^{-6}, k)) \le c\frac{\ln\ln n}{\ln n}.$$

Proof. It suffices to prove the result for $k = n$. Fix n; let $m = m_m = [(\ln n)^2]$; and choose $j_1 < j_2 < \cdots < j_m$ (depending on n) satisfying

$$n - n(\ln n)^{-6} \leq j_i \leq n, \;\; i = 1, \ldots, m,$$

$$j_i - j_{i-1} \geq 2n(\ln n)^{-9}, \;\; i = 2, \ldots, m.$$

Let $J(k, n)$ be the indicator function of

$$\{S\{k - n(\ln n)^{-9}, k] \cap S(k, k + n(\ln n)^{-9}] = \emptyset\},$$

and

$$X = X_n = \sum_{i=1}^{m} I(j_i),$$

$$\bar{X} = \bar{X}_n = \sum_{i=1}^{m} J(j_i, n).$$

By Lemma 4.1,

$$E(J(j_i, n)) = E(I(j_i))(1 + O(\frac{\ln \ln n}{\ln n})),$$

and hence

$$E(\bar{X}) = E(X)(1 + O(\frac{\ln \ln n}{\ln n})),$$

in other words,

$$E(\bar{X} - X) \leq c\frac{\ln \ln n}{\ln n} E(X). \tag{4.4}$$

We can note that

$$E(Z(n - n(\ln n)^{-6}, n)) \leq P\{X = 0\}$$

$$\leq P\{\bar{X} - X \geq \frac{1}{2}E(X)\} + P\{\bar{X} \leq \frac{1}{2}E(X)\}.$$

We can estimate the first term using (4.4),

$$P\{\bar{X} - X \geq \frac{1}{2}E(X)\} \leq 2[E(X)]^{-1}E(\bar{X} - X)$$
$$\leq c\frac{\ln\ln n}{\ln n}.$$

To estimate the second term, note that $J(j_1, n), \ldots, J(j_m, n)$ are independent and hence

$$\mathrm{Var}(\bar{X}) = \sum_{i=1}^{m} \mathrm{Var}(J(j_i, n)) \leq \sum_{i=1}^{m} E(J(j_i, n)) \leq E(\bar{X}).$$

Hence by Chebyshev's inequality, for n sufficiently large,

$$P\{\bar{X} \leq \frac{1}{2}E(X)\} \leq P\{\bar{X} \leq \frac{2}{3}E(\bar{X})\} \leq c[E(\bar{X})]^{-1}.$$

But

$$E(\bar{X}) \geq c(\ln n)^2 E(I(n)) \geq c(\ln n)^{11/8}.$$

Hence, $P\{\bar{X} \leq \frac{1}{2}E(X)\} \leq c(\ln n)^{-11/8}$ and then the lemma is proved. □

Recall from the beginning of this chapter that Y_n is the indicator function of the event "the n^{th} point is not erased" and $a_n = E(Y_n)$. Suppose that for some $0 \leq k \leq n$, loops are erased only on $S[k, \infty)$, so that S_k is considered to be origin. Let $Y_{n,k}$ be the probability that S_n is erased in this procedure. Clearly $E(Y_{n,k}) = a_{n-k}$. Now suppose $0 \leq k \leq n - n(\ln n)^{-6}$ and $Z(n - n(\ln n)^{-6}, n) = 0$. To be precise, there exists a loop-free point between $n - n(\ln n)^{-6}$ and n. Then we can check that $Y_{n,k} = Y_n$, and hence by the previous lemma,

$$P\{Y_n \neq Y_{n,k}\} \leq E(Z(n - n(\ln n)^{-6}, n)) \leq c\frac{\ln\ln n}{\ln n}.$$

Therefore, for $n(\ln n)^{-6} \leq k \leq n$,

$$|a_k - a_n| \leq c a_n (\ln n)^{-3/8},$$

i.e.,

$$a_k = a_n(1 + o((\ln n)^{-1/4})). \tag{4.5}$$

The second inequality follows from the estimate $a_n \geq f(n) \approx (\ln n)^{-1/2}$. Combining this with Theorem 2.2 we conclude

$$-\frac{1}{2} \leq \liminf_{n\to\infty} \frac{\ln a_n}{\ln \ln n} \leq \limsup_{n\to\infty} \frac{\ln a_n}{\ln \ln n} \leq -\frac{1}{3}.$$

We can also conclude

$$E(\rho(n)) \sim n a_n.$$

The following theorem [13] shows that the number of points remaining after erasing loops satisfies a weak law of large numbers.

Theorem 4.3 (Lawler). *If $d = 4$, then*

$$\frac{\rho(n)}{n a_n} \to 1, \quad \textit{in probability.}$$

Proof. For each n, choose

$$0 \leq j_0 < j_1 < \cdots < j_m = n$$

such that $(j_i - j_{i-1}) \sim n(\ln n)^{-2}$, uniformly in i. Then $m \sim (\ln n)^2$. Erase loops on each interval $[j_i, j_{i+1}]$ separately (do finite loop-erasing on $S[j_i, j_{i+1}]$). Let $\tilde{Y}_k$ be the indicator function of the event "S_k is not erased in this finite loop-erasing". Let $K_0 = [0, 0]$, and for $i = 1, \ldots, m$, let K_i be the interval

$$K_i = [j_i - n(\ln n)^{-6}, j_i].$$

Let $R_i, i = 1, \ldots, m$, be the indicator function of the component of the event, "there exists loop-free points in both K_{i-1} and K_i", i.e., the complement of the event

$$\{Z(j_{i-1} - n(\ln n)^{-6}, j_{i-1}) = Z(j_i - n(\ln n)^{-6}, j_i) = 0\}.$$

By Lemma 4.2,

$$E(R_i) \leq c\frac{\ln\ln n}{\ln n}. \tag{4.6}$$

We can note that if $j_i \leq k < j_{i+1} - n(\ln n)^{-6}$ and $R_i = 0$, then

$$Y_k = \tilde{Y}_k.$$

Therefore, for n sufficiently large,

$$\begin{aligned} |\sum_{k=0}^{n} Y_k - \sum_{k=0}^{n} \tilde{Y}_k| &\leq c[m(n(\ln n)^{-6}) + 2n(\ln n)^{-2}\sum_{i=1}^{m} R_i] \\ &\leq cn(\ln n)^{-4} + cn(\ln n)^{-2}\sum_{i=1}^{m} R_i. \end{aligned}$$

But by (4.6),

$$\begin{aligned} P\{\sum_{i=1}^{m} R_i \geq (\ln n)^{5/4}\} &\leq (\ln n)^{-5/4} E(\sum_{i=1}^{m} R_i) \\ &\leq c\ln\ln n(\ln n)^{-1/4}. \end{aligned}$$

Therefore,

$$P\{|\sum_{k=0}^{n} Y_k - \sum_{k=0}^{n} \tilde{Y}_k| \geq cn(\ln n)^{-3/4}\} \to 0.$$

Since $na_n \geq cn(\ln n)^{-5/8}$, this implies

$$(na_n)^{-1}(\sum_{k=0}^{n} Y_k - \sum_{k=0}^{n} \tilde{Y}_k) \to 0 \quad \text{in probability.} \tag{4.7}$$

We can write

$$\sum_{k=0}^{n} \tilde{Y}_k = 1 + \sum_{i=1}^{m} X_i,$$

where $X_1, \ldots, X_m$ are independent random variables,

$$X_i = \sum_{k=j_{i-1}}^{j_1 - 1} \tilde{Y}_k.$$

Notice that

$$\mathrm{Var}(X_i) \leq E(X_i^2) \leq ||X_i||_\infty E(X_i) \leq cn(\ln n)^{-2} E(X_i),$$

and hence

$$\mathrm{Var}(\sum_{k=0}^{n} \tilde{Y}_k) \leq cn(\ln n)^{-2} E(\sum_{k=0}^{n} \tilde{Y}_k).$$

Then by Chebyshev's inequality,

$$\begin{aligned} P\left\{ |\sum_{k=0}^{n} \tilde{Y}_k - E(\sum_{k=0}^{n} \tilde{Y}_n)| \geq (\ln n)^{-1/2} E(\sum_{k=0}^{n} \tilde{Y}_k) \right\} &\leq cn(\ln n)^{-1}[E(\sum_{k=0}^{n} \tilde{Y}_k)]^{-1} \\ &\leq c(\ln n)^{-3/8}. \end{aligned}$$

Which implies

$$[E(\sum_{k=0}^{n} \tilde{Y}_k)]^{-1} \sum_{k=0}^{n} \tilde{Y}_k \to 1 \quad \text{in probability.}$$

It is easy to check using (4.5) that $E(\sum_{k=0}^{n} \tilde{Y}_k) \sim na_n$ and hence by (4.7),

$$(na_n)^{-1}\rho(n) = (na_n)^{-1} \sum_{k=0}^{n} Y_k \to 1 \quad \text{in probability.} \qquad \square$$

The next theorem [13] is the main result of this chapter and shows that the LERW converges to Brownian motion in $d = 4$.

Theorem 4.4 (Main Result). *If $d = 4$, and*

$$\hat{W}_n(t) = \frac{d\sqrt{a_n}\hat{S}([nt])}{\sqrt{n}},$$

then $\hat{W}_n(t)$ converges in distribution to $B(t)$, where B is a standard Brownian motion.

Proof. Since $a_n \geq c(\ln n)^{-5/8}$, it follows from (4.5) that $a_{[n/a_n]} \sim a_n$. Therefore, by Theorem 4.3,

$$\frac{\rho([n/a_n])}{n} \to 1 \quad \text{in probability.}$$

It is not difficult using the monotonicity of ρ to show that

$$\frac{\sigma(n)a_n}{n} \to 1 \quad \text{in probability.} \tag{4.8}$$

Donsker's theorem states that if

$$W_n(t) = dn^{-1/2}S([nt]),$$

then

$$W_n(t) \to B(t),$$

in the metric space $C[0,1]$, where B is a standard Brownian motion in $\mathbb{R}^d$. Now suppose that $b_n \to \infty$ and

$$r_n(t) = \frac{\sigma([nt])}{b_n} \to t.$$

Then by tightness in $C[0,1]$ of the sequence W_n,

$$\frac{S(\sigma([nt]))}{\sqrt{b_n}} - \frac{S([b_n t])}{\sqrt{b_n}} \to 0,$$

and hence

$$\frac{dS(\sigma([nt]))}{\sqrt{b_n}} \to B(t).$$

Now fix $\epsilon > 0$ and choose $k \geq 3\epsilon^{-1}$. Let $\delta > 0$. Then by (4.8), for all n sufficiently large,

$$P\{|\frac{\sigma([nj/k])a_{[nj/k]}}{[nj/k]} - 1| \geq \frac{\epsilon}{4}\} \leq \frac{\delta}{k}, \quad j = 1, \ldots, k.$$

Since $a_{[n/k]} \sim a_n$ this implies for n sufficiently large

$$P\{|\frac{\sigma([nj/k])a_n}{n} - \frac{j}{k}| \geq \frac{\epsilon}{2}\} \leq \frac{\delta}{k}, \quad j = 1, \ldots, k.$$

But since σ is increasing and $k \geq 3\epsilon^{-1}$, this implies for n sufficiently large,

$$P\{\sup_{0 \leq t \leq 1} |\frac{\sigma([nt])a_n}{n} - t| \geq \epsilon\} \leq \delta.$$

Since this holds for any $\epsilon, \delta > 0$,

$$\frac{\sigma([nt])a_n}{n} \to t,$$

and therefore we have proved the theorem. □

Chapter 5

Scaling limit of LERW in $d = 3$

For $d < 4$, the number of points erased is not uniform from path to path and a Gaussian limit is not expected. LERW has no natural continuum equivalent; in addition, Brownian motion has a dense set of loops and therefore it is not clear how to remove them in chronological order. A paper by Kozma [11] gives some promising results to show that the scaling limit exists and is invariant under rotations and dilations.

If we denote by $L(r)$ the expected number of vertices in the LERW until it gets to a distance of r, then it was proved that

$$cr^{1+\epsilon} \leq L(r) \leq Cr^{5/3},$$

where ϵ, c and C are some positive numbers. This suggests that the scaling limit should have Hausdorff dimension between $1 + \epsilon$ and $5/3$ almost surely. Numerical experiments show that it should be 1.62 ± 0.01.

The next theorem by Kozma [11] is the main result of this chapter and shows that

LERW has a scaling limit in $d = 3$.

Theorem 5.1 (Main Result). *Let $D \subset \mathbb{R}^3$ be a polyhedron and let $a \in D$. Let $\mathbb{P}_n$ be the distribution of loop-erased random walk on $D \cap 2^{-n}\mathbb{Z}^3$ starting from a and stopped when hitting ∂D. Then $\mathbb{P}_n$ converge in the space $\mathcal{M}(\mathcal{H}(\overline{D}))$ where $\mathcal{H}(\overline{D})$ is the space of compact subsets of $\overline{D}$ with the Hausdorff metric, and $\mathcal{M}(\mathcal{H}(\overline{D}))$ is the space of measures on $\mathcal{H}(\overline{D})$ with the topology of weak convergence.*

In this paper, we will not give more details about the scaling limit of LERW in $d = 3$. For further information see [11].

Chapter 6

LERW in $d = 2$

Since simple random walk is recurrent in two dimensions, we cannot construct a two-dimensional loop-erased random walk by erasing loops from an infinite simple random walk. However, we can define the process as a limit of walks obtained from erasing loops of finite walks [13]. Let S_j be a simple random walk in $\mathbb{Z}^2$ and let

$$\xi_m = \inf\{j > 0 : |S_j| \geq m\}.$$

For any $k \leq m$, we can define a measure on Γ_k by taking simple random walks stopped at the random time ξ_m, erasing loops, and considering the first k steps. To be precise, we define $\hat{P}^m = \hat{P}^m_k$ on Γ_k by

$$\hat{P}^m(\gamma) = P\{L[S_0, \ldots, S(\xi_m)](j) = \gamma(j),\ j = 0, \ldots k\},$$

where L is the loop-erasing operation defined in Chapter 2.

Although there are three different approaches to prove the scaling limit of LERW in dimension 2 as mentioned in Chapter 1, we will focus on the connection to the stochastic Loewner evolution (SLE) [17].

In 1999, Schramm [22] introduced the stochastic Loewner evolution (and now called the Schramm-Loewner evolution) while considering loop-erased random walk. In 2002, Lawler, Schramm and Werner [17] building on early work, proved that the scaling limit of LERW in a simply connected domain $D \subsetneq \mathbb{C}$ is equal to the *radial* SLE_2 path.

In order to fully understand the concept of stochastic Loewner evolution and the scaling limit of LERW in dimension 2, it is critical to review some important concepts from complex analysis as may be found in [4] and [1].

Let i be the imaginary unit with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a complex number $\alpha + i\beta$. We say that α and β are the real and imaginary parts, respectively, of the complex number.

$D \subset \mathbb{C}$ is a *domain* if it is an open, connected set larger than a single point in the complex plane $\mathbb{C}$. Furthermore, a domain D is said to be *simply connected* if its complement is connected.

A complex-valued function $f : D \to \mathbb{C}$ of a complex variable is *differentiable* at a point $z_0 \in D$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0.

Let $f : D \to \mathbb{C}$ be a complex-valued function of a complex variable.

- If $f'(z_0)$ exists $\forall\ z_0 \in D$, then f is called *analytic* on D,

- If $f(z_0) \neq f(z_1) \;\; \forall\, z_0 \neq z_1,\, z_0,\, z_1 \in D$, then f is called *univalent* on D.

A function f is *conformal* if it is analytic and univalent on D.

We will refer to the following normalization condition for analytic functions $f : D \to \mathbb{C}$ as the hydrodynamic normalization:

$$\lim_{z\to\infty,\, z\in D} (f(z) - z) = a,$$

where $a \geq 0$ is a non-negative real number.

Let D, D' be domains. A function $f : D \to D'$ is a *conformal transformation* if it is conformal on D and onto D'. In particular, $f'(z_0) \neq 0 \;\; \forall\, z_0 \in D$, and $f^{-1} : D' \to D$ is also a conformal transformation.

The Riemann mapping theorem states that every simple connected domain can be mapped conformally onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 6.1 (Riemann mapping theorem). *Let D be a simply connected domain which is a proper subset of a complex plane. Let $z_0 \in D$ be a given point. Then there exists a unique conformal transformation $f : D \to \mathbb{D}$ satisfying*

$$f(z_0) = 0, \quad f'(z_0) > 0.$$

In addition, if D is Jordan domain, the Riemann mapping can be extended continuously to the boundary, and the extended function maps the boundary curve in one-to-one fashion onto the unit circle. This is the statement of the Carathéodory extension theorem.

Theorem 6.2 (Carathéodory extension theorem). *If D is a bounded domain and ∂D is a Jordan curve (i.e., a closed curve without self-intersections), and $f : D \to \mathbb{D}$ is a conformal transformation, then f can be extended to a homeomorphism of $D \cup \partial D$ onto $\mathbb{D} \cup \partial\mathbb{D}$.*

Note that if ∂D is "smooth" then $f'(z)$ is well defined for $z \in \partial D$.

Lévy in 1950 proved the conformal invariance of Brownian motion. This result is of fundamental importance to find the scaling limit of LERW in $d = 2$. Further details may be found in [2].

Theorem 6.3 (Conformal invariance of Brownian motion). *Let D be an open domain in the complex plane, $x \in D$, and let $f : D \to D'$ be analytic. Let $\{B_t,\, t \geq 0\}$ be a planar Brownian motion started at x and $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$ be its first exit time from the domain D. Then the process $\{f(B_t) : 0 \leq t \leq \tau_D\}$ is a time-changed Brownian motion. More precisely, there exists a planar Brownian motion $\{\tilde{B}_t, t \geq 0\}$ such that, for any $t \in [0, \tau_D)$,*

$$f(B_t) = \tilde{B}_{\sigma_t} \quad \textit{where} \quad \sigma_t = \int_0^t |f'(B_s)|^2 ds.$$

If additionally, f is conformal, then σ_{τ_D} is the first exit time from D' by $\{\tilde{B}_t, t \geq 0\}$.

6.1 Loewner's equation

In 1923, Loewner introduced the equation now bearing his name in order to prove a special case of the Bieberbach conjecture ($|a_3| \leq 3$). The Bieberbach conjecture was later established in full by deBranges—the Loewner equation turned out to be central to his argument.

There are three closely related Loewner equations: chordal which is for clusters growing from a boundary point towards a different point, radial[1] for clusters growing from a boundary point to an interior point, and whole plane which is for clusters growing from one point to infinity in $\mathbb{C}$.

We will start with the chordal Loewner equation which consists of clusters growing from a boundary point towards to a boundary point.

The *chordal Loewner equation* describes the time development of an analytic map into the upper half of the complex plane in the presence of a "forcing", a defined singularity moving around the real axis. The applications of this equation use the trace, the locus of singularities in the upper half plane.

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half complex plane, and let $\gamma : [0, \infty) \to \bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$ be a simple curve (no self intersections) with

- $\gamma(0) = 0$,
- $\gamma(0, \infty) \subseteq \mathbb{H}$,
- $\gamma(t) \to \infty$ as $t \to \infty$.

For each $t \geq 0$ suppose that $K_t := \gamma[0, t]$.

Now let $\mathbb{H}_t := \mathbb{H} \setminus K_t$ be the slit half plane which is a simply connected domain since K_t is a bounded, compact set. Then, by the Riemann mapping theorem there

[1]The radial equation is what is commonly referred to as Loewner's differential equation in the complex variable literature.

exists a conformal transformation g_t from $\mathbb{H}_t$ to $\mathbb{H}$, i.e.,

$$g_t : \mathbb{H}_t \longrightarrow \mathbb{H}$$

with $g_t(\infty) = \infty$.

Using the Schwartz reflection principle, as $z \to \infty$ we can expand g_t around ∞. Therefore,

$$g_t(z) = bz + a(0) + \frac{a(t)}{z} + O(\frac{1}{z^2}),$$

with $b > 0$ and $a(t) \in \mathbb{R}$.

Consider the expansion of $f(z) = [g_t(1/z)]^{-1}$ about the origin. f maps $\mathbb{R}$ to $\mathbb{R}$ so the coefficients in the expansion are real and $b > 0$. For convenience, we choose the unique g_t which satisfies the "hydrodynamic normalization"

$$\lim_{z\to\infty}(g_t(z) - z) = 0,$$

i.e., we choose $b = 1$, $a(0) = 0$. The constant $a(t) := a(K_t)$ only depends on the set K_t. Then $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ with $g_t(\infty) = \infty$ is

$$g_t(z) = z + \frac{a(t)}{z} + O(\frac{1}{z^2})$$

where $a(t) := a(K_t) = a(\gamma[0, t])$ is called the half-plane capacity from ∞.

We now list several facts about $a(t)$:

- $a(t) = \lim_{z\to\infty} z(g_t(z) - z)$,
- if $s < t$, then $a(s) < a(t)$,
- $s \longmapsto a(s)$ is continuous,

- $a(0) = 0$, $a(t) \to \infty$ as $t \to \infty$.

Now assume that the parametrization of $\gamma(t)$ is chosen so that $a(t) = 2t$.

Suppose $K_t := \gamma[0,t]$ with $\mathbb{H}_t := \mathbb{H} \setminus K_t$ and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map as before. If $U_t := g_t(\gamma(t))$ (the image of $\gamma(t)$), then $U_t \in \partial\mathbb{H}$ and g_t satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z,$$

which is the Loewner's equation with the identity map as initial data i.e., $g_0 : \mathbb{H} \to \mathbb{H}$.

Theorem 6.4 (Loewner, 1923). *For fixed z, $g_t(z)$ is the solution of the chordal Loewner equation is*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z. \tag{6.1}$$

We can think of this equation in two ways. Given a suitable growing curve $t \longmapsto \gamma(t)$, $t \in [0, \infty)$, we can define a sequence of conformal mappings $g_t : \mathbb{H}_t \to \mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well defined}\} = \mathbb{H} \setminus K_t$, and hence a real continuous function $U_t \equiv g_t(\text{tip})$, which will automatically satisfy the Loewner equation. Conversely, given a (suitable) continuous real function U_t (a sufficient condition is that it has Hlder exponent $\alpha > \frac{1}{2}$), we can integrate the Loewner equation to get a sequence of functions g_t and hence a sequence of tips $\gamma(t) = g_t^{-1}(U_t)$ which will trace out a curve.

Theorem 6.5 (Rohde-Marshall, 2001). *If U is "nice" (Hölder 1/2 continuous with sufficiently small Hölder 1/2 norm), then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0,t]$.*

A real-valued function U on $\mathbb{R}^d$ satisfies a Hölder condition, or is Hölder continuous, when there are non-negative real constants C, α, such that $\forall\ x,\ y \in \mathbb{R}^d$,

$$|U(x) - U(y)| \leq C|x - y|^{\alpha}.$$

This condition generalizes to functions between any two metric spaces. The number α is called the exponent of the Hölder condition. If $\alpha = 1$, then the function satisfies a Lipschitz condition. If $\alpha = 0$, then the function simply is bounded.

The *radial Loewner equation* describes the evolution of a hull from the boundary of a domain to an interior point. For ease we will consider the unit disk $\mathbb{D}$ and the interior point the origin. We will call a compact set if $K \subset \mathbb{D}\backslash\{0\}$ a hull if $K = \overline{K \cap \mathbb{D}}$ and $\mathbb{D} \setminus K$ is simply connected.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and let $\gamma[0, \infty) \to \bar{\mathbb{D}} = \mathbb{D} \cup \mathbb{R}$ be a simple path in the closed unit disk, such that

- $\gamma(0) \in \partial\mathbb{D}$,
- $\gamma(0, \infty) \subset \mathbb{D}$,
- $\gamma(\infty) = 0$.

For each $t \geq 0$ suppose that $K_t := \gamma[0, t]$ and let $\mathbb{D}_t := \mathbb{D} \setminus K_t$. By the Riemann mapping theorem there is a unique conformal homeomorphism

$$g_t : \mathbb{D}_t \to \mathbb{D}$$

satisfying the normalization $g_t(0) = 0$ and $g_t'(0) > 0$.

It turns out $g_t'(0)$ is monotone increasing (by the Schwartz lemma) in t, $g_0'(0) = 1$ and $\sup_t g_t'(0) = \infty$.

It is natural to reparametrize γ according to the "capacity" from zero, i.e.,

$$g_t'(0) = \exp(t) \ \forall \ t \geq 0.$$

Since the boundary of $\mathbb{D}_t$ is "nice" as defined before, g_t extends continuously to $\mathbb{D}_t \cup \{\gamma(t)\}$. Now set if we $U_t := g_t(\gamma(t))$, then $U_t \in \partial\mathbb{D}$ and satisfies the radial Loewner equation

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + U_t}{g_t(z) - U_t} \quad g_0(z) = z, \tag{6.2}$$

which shows that everything is encoded by U_t. We call $(U_t, t \geq 0)$ the *driving function* of the curve γ.

The driving function U is sufficient to recover the two-dimensional path $\gamma(t)$, because the process may be reversed. Suppose that we start with a continuous driving function

$$U : [0, \infty) \to \partial\mathbb{D}.$$

Then for every $z \in \bar{\mathbb{D}}$ there is a solution $g_t(z)$ of (6.2) with initial value $g_0(z) = z$ up to some time $\tau(z) \in (0, \infty]$, beyond which a solution does not exist. That is the only possible reason why (6.2) cannot be solved beyond time $\tau(z)$. Then we define

$$K_t := \{z \in \bar{\mathbb{D}} : \tau(z) \leq t\} \tag{6.3}$$

and

$$\mathbb{D}_t := \mathbb{D} \setminus K_t \tag{6.4}$$

is the domain of definition of g_t. The set K_t is called the hull at time t. If U arises from a simple path γ as described before, then we can recover γ from U by letting $\gamma(t) = g_t^{-1}(U_t)$. However, if $U : [0, \infty) \to \mathbb{D}$ is an arbitrary continuous driving function, then K_t need not to be a path, and even if it is a path, it is not necessarily a simple path.

The *whole plane Loewner equation* describes the evolution of hulls starting a with a singles point, which we choose to be the origin, going to infinity. We will not give a detailed explanation, but roughly we can consider the whole plane Loewner equation as the radial equation starting at $t = -\infty$. The hulls starts with only the origin, immediately grows a small amount so that it has positive capacity and then evolves as a radial Loewner process going to infinity.

For the purpose of this paper we will be focused on the radial Loewner equation. Further details about the Loewner equation from a complex analysis point-of-view may be found in [4]. For details about the Loewner equation from a probabilistic point-of-view (and its connections to SLE as presented in the next section) consult [16].

6.2 Radial stochastic Loewner evolution (SLE)

The radial stochastic Loewner evolution (denoted radial SLE_κ) is a random growth process based on the radial Loewner equation with a one-dimensional Brownian motion running with speed κ as the driving process.

Formally, the radial stochastic Loewner evolution with parameter κ is the process $\{\gamma(t), t \geq 0\}$ where the driving function U_t is set to be $U_t = \exp(iB_{\kappa t})$ and $B : [0, \infty) \to \mathbb{R}$ is Brownian motion.

To be precise, let B_t be Brownian motion on the unit circle $\partial\mathbb{D}$ started from a uniform-random point B_0 and set $U_t := B_{\kappa t}$. (We can take the point B_0 to be uniformly distributed in $[0, 2\pi]$.) The conformal maps g_t are defined as the solution of the radial Loewner equation

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + U_t}{g_t(z) - U_t} \quad g_0(z) = z,$$

for $z \in \mathbb{D}$. The sets K_t and $\mathbb{D}_t$ are defined as in (6.3) and (6.4).

It is important to remark that although varying the parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

The result of Rohde and Schramm (2001) [21] allows us to define γ from the boundary to the origin (if $\kappa \neq 8$). Furthermore, for different values of κ we have the following result. With probability one:

- if $0 < \kappa \leq 4$, then $\gamma(t)$ is a random simple curve avoiding $\mathbb{R}$,
- if $4 < \kappa < 8$, then $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself. In addition, these paths hit $\mathbb{R}$.
- If $\kappa \geq 8$, then $\gamma(t)$ is a space filling curve. It has double points, but does not cross itself.

6.3 Convergence of LERW to SLE_2 in $d = 2$

As already noted, it was Schramm who introduced this one-parameter family of random growth processes based on Loewner's differential equation in which the driving term U_t is a time-scaled one-dimensional Brownian motion. This process is called the stochastic (or Schramm–)Loewner evolution SLE_κ where the parameter κ is the time scaling constant for the driving Brownian motion.

It was originally conjectured that the scaling limit of LERW was SLE_2, and this conjecture was proved to be equivalent to the conformality of the LERW scaling limit [22].

In 2002 Lawler, Schramm and Werner [17] established this conjecture and showed that the scaling limit of LERW in a simply connected domain $D \subsetneq \mathbb{C}$ is equal to radial SLE_2 path.

The convergence of LERW to SLE_2 is derived as a consequence of three facts [24]:

- The "Markovian property" holds in the discrete case.
- Some macroscopic functionals of the model converge to conformally invariant quantities in the scaling limit (for a wide class of domains).
- One has "a priori" bounds on the regularity of the discrete paths.

As stated in [17], let $D \subsetneq \mathbb{C}$ be a simply connected domain with $0 \in D$ and let $D_\delta = D \cap \delta\mathbb{Z}^2$ be the grid in D. Consider γ^δ the time reversal loop-erasure of simple random walk in D_δ started at 0 and stopped when it hits ∂D_δ. Let γ denote the radial SLE_2 in the unit disk $\mathbb{D}$ started uniformly on the unit circle and aiming at 0.

Let Φ be the conformal map from $\mathbb{D}$ onto D that preserves 0. When the boundary of D is very rough, the conformal map Φ might not be extended continuously to the boundary, but it has been proved that even in this case the image of SLE_2 path has a unique endpoint on ∂D.

We endow the set of paths with the metric of uniform convergence modulo time-reparametrization

$$d(\Gamma, \Gamma') = \inf_{\varphi} \sup_{t \geq 0} |\Gamma(t) - \Gamma'(\varphi(t))|,$$

where infimum is over all increasing bijections φ from $[0, \infty)$ into itself.

Theorem 6.6 (Main Result). *The law of γ^δ converges in distribution to the law of $\Phi(\gamma)$ as $\delta \to 0$ with respect to the metric on the space of curves.*

More recently, a number of other works have been published which explore the relationship between LERW and SLE_2. In particular, the work of Lawler and Werner [19] on the Brownian loop soup, and of Lawler and Trujillo Ferreras [18] on the random walk loop soup indicate that SLE_2 may (in a sense) be viewed as "loop-erased" Brownian motion. Kozdron and Lawler have established that the scaling limit of Fomin's identity for loop-erased random walk may be given in terms of SLE_2. See [9] and the references therein for more details.

Chapter 7

Wilson's algorithm

LERW was introduced by Lawler in 1980 [12] in an attempt to analyze self-avoiding walk (SAW). Eventually, it was discovered that SAW and LERW are in different universality classes.

Basically, LERW existed as an interesting mathematical idea without any applicability until 1996, when Wilson discovered its use in finding spanning trees of graphs in an efficient way [25].

Let $G = (V, E)$ be a graph. A spanning tree of G is a subgraph of G containing all vertices and some of the edges which is a tree, i.e., connected and with no cycles. A uniform spanning tree (UST) is a (random) spanning tree chosen among all the possible spanning trees of G with equal probability (uniformly).

Now let v and w be two vertices in G. Any spanning tree contains precisely one simple (non-intersecting) path between v and w. Taking this path in a uniform spanning tree gives a random simple path. It turns out that the distribution of this path is identical to the distribution of the loop-erased random walk starting at v and stopped at w.

As mentioned in Chapter 2, the distribution of the LERW starting at v and stopped at w is identical to the distribution of the reversal of LERW starting at w and stopped at v. This is not a trivial fact at all since loop-erasing a path and the reverse path do not give the same result. It is only the distributions that are identical.

Sampling a UST is complicated. Even a relatively modest graph (100×100 grid) has way too many spanning trees to allow one to enumerate a complete list.

Wilson [25] established an algorithm to generate USTs using LERW.

Take any two vertices and perform loop-erased random walk from one to the other. Now take a third vertex (not on the constructed path) and perform loop-erased random walk until hitting the already constructed path. This gives a tree with three leaves. Choose a fourth vertex and do loop-erased random walk until hitting this tree. Continue until the tree spans all the vertices.

More precisely, Wilson's algorithm runs as follows:

- pick an arbitrary ordering $v_0, v_1, \ldots, v_m$ for the vertices in G;
- let $T_0 = \{v_0\}$;
- inductively, for $n = 1, 2, \ldots, m$, define T_n to be the union of T_{n-1} and a conditionally independent LERW path from w_n to T_{n-1};
- if $w_n \in T_{n-1}$, then $T_n = T_{n-1}$.

Regardless of the chosen order of the vertices, T_m is a uniform spanning tree on G.

Wilson's algorithm gives a natural extension of the definition of UST to infinite recurrent graphs.

In 2004, Schramm, Lawler and Werner [17] showed that the scaling limit of UST is equal to the chordal SLE_8. In particular, they proved that the limit of UST exists and is conformally invariant.

Bibliography

[1] L. V. Ahlfors. *Complex Analysis*. McGraw-Hill, New York, NY, third edition, 1979.

[2] R. F. Bass. *Probabilistic Techniques in Analysis*. Springer-Verlag, New York, NY, 1995.

[3] L. Breiman. *Probability*, volume 7 of *Classics in Applied Mathematics*. SIAM, Philadelphia, PA, 1992.

[4] P. L. Duren. *Univalent Functions*, volume 259 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, NY, 1983.

[5] A. J. Guttmann and R. J. Bursill. Critical exponents for the loop erased self-avoiding walk by Monte Carlo methods. *J. Stat. Phys.*, 59:1–9, 1990.

[6] J. Jacod and P. Protter. *Probability Essentials*. Springer-Verlag, Berlin, Germany, second edition, 2003.

[7] R. Kenyon. The asymptotic determinant of the discrete Laplacian. *Acta Math.*, 185:239–286, 2000.

[8] R. Kenyon. Long-range properties of spanning trees. *J. Math. Phys.*, 41:1338–1363, 2000.

[9] M. J. Kozdron. The scaling limit of Fomin's identity for two paths. Preprint, 2007. Available online at `arxiv.org/abs/math.PR/0703615`.

[10] G. Kozma. Scaling limit of loop erased random walk — a naive approach. Preprint, 2002. Available online at `arxiv.org/abs/math.PR/0212338`.

[11] G. Kozma. The scaling limit of loop-erased random walk in three dimensions. Preprint, 2005. Available online at `arxiv.org/abs/math.PR/0508344`.

[12] G. F. Lawler. A self-avoiding random walk. *Duke Math. J.*, 47:655–693, 1980.

[13] G. F. Lawler. *Intersections of Random Walks.* Birkhäuser, Boston, MA, 1991.

[14] G. F. Lawler. The logarithmic correction for loop-erased walk in four dimensions. In J. J. Benedetto, editor, *Proceedings of the Conference in Honour of Jean-Pierre Kahane: Special Issue of the Journal of Fourier Analysis and Applications*, pages 347–362. CRC Press, Boca Raton, FL, 1995.

[15] G. F. Lawler. Loop-erased random walk. In R. Durrett and M. Bramson, editors, *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*, volume 44 of *Progress in Probability*, pages 197–217. Birkhäuser, Boston, MA, 1999.

[16] G. F. Lawler. *Conformally Invariant Processes in the Plane*, volume 114 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[17] G. F. Lawler, O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32:939–995, 2004.

[18] G. F. Lawler and J. A. Trujillo Ferreras. Random walk loop soup. *Trans. Amer. Math. Soc.*, 359:767–787, 2007.

[19] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128:565–588, 2004.

[20] N. Madras and G. Slade. *The Self-Avoiding Walk.* Birkhäuser, Boston, MA, 1996.

[21] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. Math.*, 161:883–924, 2005.

[22] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[23] G. Slade. *The Lace Expansion and its Applications*, volume 1879 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Germany, 2006.

[24] W. Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on Probability Theory and Statistics: Ecole d'Eté de Probabilités de Saint-Flour XXXII-2002*, volume 1840 of *Lecture Notes in Mathematics*, pages 113–196. Springer-Verlag, Berlin, 2004.

[25] D. B. Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 296–303. ACM Press, Philadelphia, PA, 1996.